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Assignment 5 Problem 4.1.12
http://calclab.math.tamu.edu/~b-young/4_1_12.pdf

Problem Statement: Test the given set of vectors for linear independence:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

Solution: In order to determine whether a set of vectors is linearly independent, we go to the definition of linear independence, that is that no nontrivial linear combination of the set equals zero. It is simpler to check for linear depedence, thus:

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 = \vec{0}$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 3 \\ 4 & 3 & 3 & 2 \\ 3 & 4 & 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This can be solved simply using Gaussian Elimination. To make the process simpler, the augmented matrix needs to only be reduced to lower triangular form. This will show only the trivial case where a, b, c, and d are zero allows the above equation to be true, thus showing the vectors to be independent. The steps are shown here:

$$\begin{pmatrix} 1 & 2 & 1 & 4 & 0 \\ 2 & 1 & 2 & 3 & 0 \\ 4 & 3 & 3 & 2 & 0 \\ 3 & 4 & 4 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & -3 & 0 & -5 & 0 \\ 0 & -5 & -1 & -14 & 0 \\ 0 & -2 & 1 & -11 & 0 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & -3 & 0 & -5 & 0 \\ 0 & 0 & -3 & -17 & 0 \\ 0 & 0 & 3 & -23 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 4 & 0 \\ 0 & -3 & 0 & -5 & 0 \\ 0 & 0 & -3 & -17 & 0 \\ 0 & 0 & 0 & -40 & 0 \end{pmatrix}$$

Since the augmented matrix can be reduced to lower triangular form, the vectors are linearly independent. Being able to reduce it shows that a, b, c, and d must be zero for the combination of the vectors to be zero, thus showing that they are not dependent.

This problem can also be solved by taking the determinate of the matrix composed of the set of vectors. However, for a four-dimensional matrix, there are many chances for making simple arithmetic errors when working by hand. For this reason the above method was used. As a check however, the computer was used to calculate the determinate, which is -40. Since the determinate is not zero, the matrix is nonsingular, and thus the vectors are linearly independent, as shown above.