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Assignment 5 Problem 4.4.13
http://calclab.math.tamu.edu/~b-young/4_4_13.pdf

Problem Statement: Find the matrix (with respect to standard bases) of the linear operator $L\colon \mathcal{P}_1 \to \mathcal{P}_2$ defined by

$$L(p)(t) = \int_0^t p(\tau)\, d\tau + p'(t).$$

Solution: First the bases of the function must be defined. The natural basis for $\mathcal{P}_1$ is $\{t, 1\}$ and the natural basis for $\mathcal{P}_2$ is $\{t^2, t, 1\}$, where $p(t)$ is in $\mathcal{P}_1$ and $L(p)(t)$ is in $\mathcal{P}_2$.

The matrix is defined as that matrix C which makes $L(p)(t) = Cp(t)$ and can be found by applying L to each of the components of the basis for $\mathcal{P}_1$ and placing these in columns. That is:

$$\begin{matrix} t^2 \to \\ t \to \\ 1 \to \end{matrix} \begin{pmatrix} L(t) & L(1) \\ | & | \\ \perp & \perp \end{pmatrix} = C$$

Place t and 1 into L to find these values and place them into the matrix:

$$L(t) = \frac{t^2}{2} + 1$$
$$L(1) = t$$

Thus,

$$C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This can easily be checked by substituting an arbitrary $\mathcal{P}_1$ polynomial into both forms, the standard method and the matrix derived above. For example:

$$L(3t+5) = \frac{3}{2}t^2 + 5t + 3$$

$$L(3t+5) = Cp(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 5 \\ 3 \end{pmatrix}$$

Which are equivalent results.