Xujia Zhang Account Name: x-zhang email: x-zhang@calclab.math.tamu.edu
Assignment (week) 3 Problem: 3.2.15
URL: http://people.tamu.edu/~x0z0680/hw3a(revised).pdf Revised Version

3.2.15 Let D: $C^1 \rightarrow C$ be the differentiation operator, and let $T: C \rightarrow C$ be the operation of multiplication of a function (f(t)) by its variable, t. (The domain of the functions f may be **R** or any fixed subinterval of **R**.)

(a) Check that $C \equiv DT-TD$ makes sense as a linear function from C^1 into C, provided that the domain of each operator is properly interpreted. (C is called the commutator of the two original operators.)

Solution:

Because the domains and ranges of the linear functions match up, and the domain of each operator is properly interpreted, therefore, the composite functions are defined:
Domain and range of D are any polynomials with any power
Domain and range of T are any polynomials with any power
Domain of f(t) is R

Two conditions need to be met in order to prove linearity:

L[x+y] = L[x] + L[y]

L[c x] = c L[x]

However, these two equations in the definition of linearity can be summarized in one equation: L[rx+y]=rL[x]+L[y], where x, y, and L are all vectors.
Thus, we need to prove C(rx(t)+y(t))=rC(x(t))+C(y(t))

→

C(rx(t)+y(t))=DT(rx(t)+y(t))-TD(rx(t)+y(t))

=D{rtx(t)+ty(t)}-T{rx'(t)+y'(t)}

={rx(t)+rtx'(t)+y(t)+ty'(t)}-{rtx'(t)+ty'(t)}

=rx(t)+rtx'(t)+y(t)+ty'(t)-rtx'(t)-ty'(t)

=rx(t)+y(t)

rC(x(t))+C(y(t))=r{DT(x(t)-TD(x(t)))}+{DT(y(t))-TD(y(t))}

=r{D(tx(t))-T(x'(t))}+{D(ty(t))-T(y'(t))}

=rx(t)+rtx'(t)-rtx'(t)+y(t)+ty'(t)-ty'(t)

=rx(t)+y(t)

→

C(rx(t)+y(t))=rC(x(t))+C(y(t))
Therefore, $C \equiv DT-TD$ makes sense as a linear function from C^1 into C.

(b) Calculate C(f)(t) for $f(t)=t^2+3t$.

<u>Solution:</u>

$$
\begin{aligned}
C(f)(t) &= DT(t^2+3t)-TD(t^2+3t) \\
&= D(t^3+3t^2)-T(2t+3) \\
&= 3t^2+6t-2t^2-3t \\
&= t^2+3t
\end{aligned}
$$

Therefore, if $f(t)=t^2+3t$, $C(f)(t)= t^2+3t$.

(c) Calculate C(f)(t) for f(t)=sin(t).

<u>Solution:</u>

$$
\begin{aligned}
C(f)(t) &= DT(\sin(t))-TD(\sin(t)) \\
&= D(t\sin(t))-T(\cos(t)) \\
&= \sin(t)+t\cos t(t)-t\cos t(t) \\
&= \sin(t)
\end{aligned}
$$

Therefore, if f(t)=sin(t), C(f)(t)=sin(t).

(d) Calculate C(f)(t) for an arbitrary function $f \in C^1$.

<u>Solution:</u>

$$
\begin{aligned}
C(f)(t) &= DT(f(t))-TD(f(t)) \\
&= D(tf(t))-T(f'(t)) \\
&= f(t)+tf'(t)-tf'(t) \\
&= f(t)
\end{aligned}
$$

The above proof shows that the commutator (C) of the two original operators (D and T) will provide a result that is the same as the original function f(t).