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Assignment (week) 11 Problem: 7.1.12
URL: http://people.tamu.edu/~x0z0680/hw11(revised).pdf Revised Version

7.1.12 Let x_1 and x_2 be two real numbers. Find a quadrature (numerical integration) formula for the form:

$$\int_0^1 f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

by requiring that the formula give the right answer for f(x)=1 and for f(x)=x. Then verify that your answer reduces to the usual trapezoidal rule when $x_1 = 0$ and $x_2 = 1$.

Solution:

We are deriving a formula for approximating $\int_0^1 f(x)dx$ in terms of the values of f at 2 points. Since we require that the formula gives the exact answers for the two functions for f(x)=1 and for f(x)=x, we can solve for the coefficients c_1 and c_2. Since both the formula and the integral itself are linear functionals of f, the resulting formula will give the exact answer for any linear combination of the two basis functions.

$$1 = \int_0^1 1dx = c_1 + c_2$$

$$\frac{1}{2} = \int_0^1 xdx = c_1 x_1 + c_2 x_2$$

Solve these equations by Cramer's rule; the determinant that appears in the denominator is then:

$$V \equiv \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}$$

$$\det V = x_2 - x_1$$

$$c_1 = \frac{1}{x_2 - x_1}\begin{vmatrix} 1 & 1 \\ \frac{1}{2} & x_2 \end{vmatrix} = \frac{x_2 - \frac{1}{2}}{x_2 - x_1}$$

$$c_2 = \frac{1}{x_2 - x_1}\begin{vmatrix} 1 & 1 \\ x_1 & \frac{1}{2} \end{vmatrix} = \frac{\frac{1}{2} - x_1}{x_2 - x_1}$$

When $x_1 = 0$ and $x_2 = 1$, we have V= $x_2 - x_1 = 1$

$$c_1 = \frac{1-0.5}{1-0} = 0.5$$

$$c_2 = \frac{0.5-0}{1-0} = 0.5$$

So, the proposed formula is:

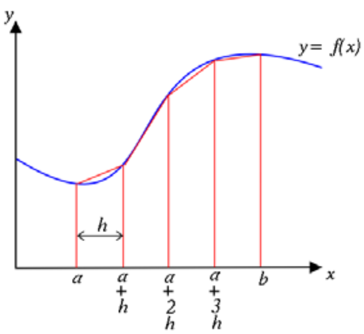
$$\int_0^1 f(x)dx \approx \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2)$$

Obviously, this is the trapezoidal rule, or at least the basic special case of it.

Here are some background information about the trapezoidal rule and the proof of why the above equation is the trapezoidal rule.
Because the trapezoidal rule is a method for finding an approximate value for a definite integral, suppose we have:

$$\int_a^b f(x)dx$$

First the area under the curve



$y = f(x)$ is divided into n strips, each of equal width $h = \frac{b-a}{n}$.

The shape of each strip is approximated to be like that of a trapezium. Hence the area of the first strip is approximately

$$\frac{h}{2}(f(a) + f(a+h))$$

similarly we approximate the area of each i strip to be

$$\frac{h}{2}(f(a+(i-1)h) + f(a+ih))$$

Adding up these areas gives us an approximate value for our definite integral:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{h}{2}(f(a+(i-1)h)+f(a+ih))$$

The equation we had was just a simplified form of this complicated equation, because in this case:

$h=1$

$a+(i-1)h = x_1$ Because x_1 is the old point; and x_2 is our new point in our case.

$a+ih = x_2$

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{h}{2}(f(a+(i-1)h)+f(a+ih)) = \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2)$$

They are the same.