Xujia Zhang Account Name: x-zhang email: x-zhang@calclab.math.tamu.edu
Assignment (week) 12 Problem: 7.5.1
URL: http://people.tamu.edu/~x0z0680/hw12b.pdf

7.5.1 Write out Gauss's theorem in two dimensions and prove it equivalent to Green's theorem. HINT: Consider two vector fields, related so that the dot product of one with the tangent vector to the boundary is equal to the dot product of the other with the normal vector to the boundary.

Solution:

Gauss's theorem is stated here: In $\Re^3$, let $\vec{A}(\vec{r})$ be a vector field, S be a closed surface, and V be the region interior to S. The normal vector on S points outward.

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V (\nabla\cdot\vec{A})d^3r = \iint_S \vec{A}\cdot d\vec{S} = \iiint_V (\nabla\cdot\vec{A})d^3r = \iiint_V div\vec{A}(x,y,z)dV$$

where $d^3r = d^3x = dV = dxdydz$.

The Green's theorem states: in $\Re^3$, let $\vec{A}(\vec{r})$ be a vector field, D be a region, C be the boundary of D (oriented so that D is on the left as on traverses C)

$$\oint_C \vec{A}\cdot dr = \iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dxdy$$

Let the boundary of D (C) to be represented by r which is a function of x and y: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ and t is bounded by two constants a and b: $a \le t \le b$.

We have two vector fields, related so that the dot product of one with the tangent vector to the boundary is equal to the dot product of the other with the normal vector to the boundary.

According to this hint, we will need to find the normal vector and the tangent vector first:

Normal vector= $\vec{n}(t) = \frac{y'(t)}{\left\|\vec{r}'(t)\right\|}\hat{i} - \frac{x'(t)}{\left\|\vec{r}'(t)\right\|}\hat{j}$

And

The Tangent vector= $\vec{T}(t) = \frac{x'(t)}{\left\|\vec{r}'(t)\right\|}\hat{i} + \frac{y'(t)}{\left\|\vec{r}'(t)\right\|}\hat{j}$

Do the calculations according to the hint with the normal vector and tangent vector substituted in.

$$\int_C \vec{A}\cdot\vec{n}ds = \int_a^b (\vec{A}\cdot\vec{n})(t)\left\|\vec{r}'(t)\right\|dt$$

$$= \int_a^b [\frac{A_x(x(t),y(t))y'(t)}{\left\|\vec{r}'(t)\right\|} - \frac{A_y(x(t),y(t))x'(t)}{\left\|\vec{r}'(t)\right\|}]\left\|\vec{r}'(t)\right\|dt$$

$$= \int_a^b A_x(x(t),y(t))y'(t)dt - \int_a^b A_y(x(t),y(t))x'(t)dt$$

$$= \int_C A_x dy - A_y dx$$

$$\Rightarrow$$

$$\int_C \vec{A}\cdot\vec{n}ds = \iint_D (\frac{\delta A_y}{\delta y} + \frac{\delta A_x}{\delta x})dA = \iint_D div\vec{A}(x,y)dA$$

Now, we compare the derivated Green's theorem and the Gauss's theorem.

$\oint_S \vec{A}\cdot d\vec{S} = \iint_S \vec{A}\cdot\vec{n}ds = \iiint_V div\vec{A}(x,y,z)dV$ ---------Gauss's theorem

$\int_C \vec{A}\cdot\vec{n}ds = \iint_D div\vec{A}(x,y)dA$ ----------------Green's theorem

Compare the two equations above; we can see that if we add one more dimension (to change from 2 dimensions to 3 dimensions) to the green's theorem, it will be equivalent to the Gauss's theorem. Or as what we did, we wrote the Gauss's theorem in two dimensions, and that's the same as the green's theorem.