12.3.5

Question:

Determine analytic formulas for u(x,t) if

$$u(x,0) = f(x) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \begin{matrix} 1 \; |x|<h \\ 0 \; |x|>h \end{matrix}$$

(Hint: Using characteristics as sketched in Fig.12.3.3, show there are two distinct regions t < h/c and t > h/c. In each, show that the solution has five different forms, depending on x.)

Solution:

The text provides the equations

$$\text{F(x)} = \begin{matrix} \frac{h}{2c}\,, x < -h \\ -\frac{x}{2c}\,, -h < x < h \\ -\frac{h}{2c}\,, x > h \end{matrix}$$

$$\text{and G(x)} = \begin{matrix} -\frac{h}{2c}\,, x < -h \\ \frac{x}{2c}\,, -h < x < h \\ \frac{h}{2c}\,, x > h \end{matrix}$$

This can then be used to find F(x) and G(x) at different times:

$$\text{F(x)} = \begin{matrix} \frac{h}{2c}\,, x - ct < -h \\ -\frac{x-ct}{2c}\,, -h < x - ct < h \\ -\frac{h}{2c}\,, x - ct > h \end{matrix}$$

$$\text{and G(x)} = \begin{matrix} -\frac{h}{2c}\,, x + ct < -h \\ \frac{x+ct}{2c}\,, -h < x + ct < h \\ \frac{h}{2c}\,, x + ct > h \end{matrix}$$

Then because $u(x,t) = F(x-ct) + G(x+ct)$we can add the two equations to find different forms in each region:

$$t < \frac{h}{c} : u(x,t) = F(x-ct) + G(x+ct) = \begin{array}{l} 0, x < -h-ct \\ \frac{h+x+ct}{2c}, -h-ct < x < -h+ct \\ t, -h+ct < x < h-ct \\ \frac{h-x+ct}{2c}, h-ct < x < h+ct \\ 0, h+ct < x \end{array}$$

$$t > \frac{h}{c} : u(x,t) = F(x-ct) + G(x+ct) = \begin{array}{l} 0, x < -h-ct \\ \frac{h+x+ct}{2c}, -h-ct < x < h-ct \\ \frac{h}{c}, h-ct < x < -h+ct \\ \frac{h-x+ct}{2c}, -h+ct < x < h+ct \\ 0, h+ct < x \end{array}$$