12.4.1

Question

Solve by the method of characteristics:

$$\frac{d^2u}{dt^2} = c^2 \frac{d^2u}{dx^2}, \text{ x>0}$$

subject to u(x,0) = 0, $\frac{du}{dt}(x,0) = 0$, and u(0,t)=h(t)

Solution:

The general solution of a wave equation is given by 12.4.5

u(x,t)=F(x-ct)+G(x+ct)

from 12.4.2 and 12.4.3

we know that u(x,0)=f(x)=0

$$\frac{du}{dt}(x,0) = g(x) = 0$$

therefore, from 12.4.6 and 12.4.7

we obtain G(x)=0, F(x)=0 only if x>0

when u(0,t)=h(t) if t>0, we obtain

h(t)=u(0,t)=F(-ct)+G(ct) if t>0 from the general solution

case(1): If x>ct, G(x+ct) and F(x-ct) only requires positive arguments, hence x>0

so we can obtain that u(x,t)=0 if x>ct

case(2): If x<ct, F(-ct) requires negative arguments

from h(t)=F(-ct)+G(ct), let z = -ct

$h(-\frac{z}{c})$ = F(z)+G(-z), F(z)=$h(-\frac{z}{c})$-G(-z) for z<0

thus, F(x-ct) = $h(\frac{x-ct}{-c})$ – G(ct-x) = $h(t-\frac{x}{c})$ – G(ct-x)

for solution x<ct is

u(x,t) = F(x-ct) + G(x+ct)

= $h(t-\frac{x}{c})$-G(ct-x)+G(x+ct)

since when x<ct, both arguments of G are positive, hence both G = 0

so we can obtain u(x,t) = $h(t-\frac{x}{c})$ if x<ct

so the general solution of this problem is in the from of

$$u(x,t) = \begin{cases} 0, & \text{if } x>ct \\ h(t-\frac{x}{c}), & \text{if } x<ct \end{cases}$$

12.3.5

Question: Determine analytic formulas for u(x,t) if

u(x,0) = f(x) = 0

$\frac{du}{dt}$(x,0) = g(x) = { 1, |x|<h

{ 0, |x|>h

from text, Pg 548, given that

G(x) = { $-\frac{h}{2c}$, x<-h

{ $\frac{x}{2c}$, -h<x<h

{ $\frac{h}{2c}$, x>h

and

F(x) = { $\frac{h}{2c}$, x<-h

{ -$\frac{x}{2c}$, -h<x<h

{ $-\frac{h}{2c}$, x>h

so we use the above froms at x=> x-ct or x+ct

G(x) = { $-\frac{h}{2c}$, x+ct<-h

{$\frac{x+ct}{2c}$, -h<x+ct<h

{ $\frac{h}{2c}$, x+ct>h

and

F(x) = { $\frac{h}{2c}$, x-ct<-h

{-$\frac{x+ct}{2c}$, -h<x-ct<h

{ $\frac{h}{2c}$, x-ct>h

since u(x,t) = F(x-ct) +G(x+ct) from 12.3.4

we can obtain 5 different forms from each regions by adding above two equations

case(1): for t<$\frac{h}{c}$, u(x,t) = F(x-ct)+G(x+ct)={ 0, x<-h-ct

{ $\frac{h+x+ct}{2c}$, -h-ct<x<-h+ct

{ t, -h+ct<x<h+ct

{ $\frac{h-x+ct}{2c}$, h-ct<x<h+ct

{ 0, x>h+ct

case(2): for t>$\frac{h}{c}$, u(x,t) = F(x-ct)+G(x+ct) ={ 0, x<-h-ct

{ $\frac{h+x+ct}{2c}$, -h-ct<x<h-ct

{ $\frac{h}{c}$, h-ct<x<-h+ct

{ $\frac{h-x+ct}{2c}$, -h+ct<x<h+ct

{ 0, x>h+ct