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Week 4 2.5.1a

Given the boundary conditions $\frac{\partial u}{\partial x}(0,y)=0,\ \frac{\partial u}{\partial x}(L,y)=0,\ u(x,0)=0,\ u(x,H)=f(x)$ of a rectangle of the dimensions $0\le x\le L,\ 0\le y\le H$, the Laplace equation can be solved. The Laplace equation is: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. Using separation of variables, this equation becomes:

$$-X''Y-XY''=0$$

$$-\frac{X''}{X}=\frac{Y''}{Y}=\lambda \quad\Rightarrow\quad \begin{array}{c}-X''=\lambda X\\ Y''=\lambda Y\end{array}$$

This means that

$$X=\cos\left(\frac{n\pi x}{L}\right)\quad\Rightarrow\quad \lambda=\frac{n^2\pi^2}{L^2} \qquad \text{for } n=0,1,2...$$

$$Y''=\frac{n^2\pi^2}{L^2}y$$

now, for n=0

$$Y=c_1+c_2y \qquad\Rightarrow\qquad Y(0)=0\ \rightarrow\ c_1=0$$

and for n≠0

$$Y=c_1\cosh\left(\frac{n\pi y}{L}\right)+c_2\sinh\left(\frac{n\pi y}{L}\right)\qquad\Rightarrow\qquad Y(0)=0\ \rightarrow\ c_1=0$$

Superposition gives

$$u(x,y)=A_0y+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

and the non-homogeneous boundary conditions gives

$$u(x,H)=f(x)=A_0H+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)$$

where

$$A_0H=\frac{1}{L}\int_0^L f(x)dx \quad\rightarrow\quad A_0=\frac{1}{LH}\int_0^L f(x)dx$$

$$A_n\sinh\left(\frac{n\pi H}{L}\right)=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad\rightarrow\quad A_n=\frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$