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Week 9
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wthrgal
5.4.6

Consider the vibrations of a nonuniform string of mass density $\rho_0(x)$. Suppose that the left end at $x=0$ is fixed and the right end obeys the elastic BC: $\frac{\partial u}{\partial x} = -\left(\frac{k}{T_0}\right)u$ at $x = L$. Suppose that the string is initially at rest with a known initial position $f(x)$. Solve this initial value problem.

From this problem statement, we know:

$$PDE: \quad \rho_0 \frac{\partial^{2u}}{\partial t^2} = T_0 \frac{\partial^{2u}}{\partial x^2}$$

$$BC: \quad u(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = -\left(\frac{k}{T_0}\right)u$$

$$IC: \quad u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

Using separation of variables, where $u(x,t) = \phi(x)h(t)$, we get:

$$\rho_0 \phi h'' = T_0 \phi'' h \quad \Rightarrow \quad \frac{h''}{h} = \frac{T_0}{\rho_0}\frac{\phi''}{\phi} = -\lambda \quad \Rightarrow \quad \begin{matrix} h'' = -\lambda h & T_0\phi'' = -\lambda \rho_0 \phi \\ h'' + \lambda h = 0 & T_0 \phi'' + \lambda \rho_0 \phi = 0 \end{matrix}$$

The boundary conditions are of Sturm-Liouville type, so ϕ_n is orthogonal with $\rho_0(x)$. Also, the eigenvalues are λ_n. The time-dependent equation has the solutions $\sin(\sqrt{\lambda_n}t)$ and $\cos(\sqrt{\lambda_n}t)$. However, since the string is initially at rest, only the cosines remain. Then, by superposition:

$$\therefore u(x,t) = \sum_{n=1}^{\infty} A_n \phi_n(x) \cos(\sqrt{\lambda_n}t)$$

The initial position gives

$$f(x) = \sum_{n=1}^{\infty} A_n \phi_n(x)$$

Then

$$A_n = \frac{\int_0^L f(x)\phi_n(x)\rho_0(x)\,dx}{\int_0^L \phi_n(x)^2 \rho_0(x)\,dx}$$