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Week 11 7.7.1
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Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \left[= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right]$$

with $u(a,\theta,t) = 0$**,** $u(r,\theta,0) = 0$**, and** $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$**.**

(7.7.45) represents the product solutions from separation of variables $\left(u(r,\theta,t) = f(r)g(\theta)h(t)\right)$:

$$J_m\left(\sqrt{\lambda_{mn}}\, r\right) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos\left(c\sqrt{\lambda_{mn}}\, t\right) \\ \sin\left(c\sqrt{\lambda_{mn}}\, t\right) \end{Bmatrix}$$

The first initial condition, $u(r,\theta,0) = 0$, implies that all the terms containing $\cos\left(c\sqrt{\lambda_{mn}}\, t\right)$ vanish. From the second initial condition, $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$, implies that the only term that is necessary is when m=3, which means only $\sin 3\theta$ the terms. Then, by superposition,

$$\boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} A_n J_3\left(\sqrt{\lambda_{3n}}\, r\right) \sin 3\theta \sin\left(c\sqrt{\lambda_{3n}}\, t\right)}$$

From the second initial condition,

$$\alpha(r)\sin 3\theta = \sum_{n=1}^{\infty} A_n \left(c\sqrt{\lambda_{3n}}\right) J_3\left(\sqrt{\lambda_{3n}}\, r\right) \sin 3\theta$$

Cancelling the $\sin 3\theta$, we find that

$$\boxed{\alpha(r) = \sum_{n=1}^{\infty} A_n \left(c\sqrt{\lambda_{3n}}\right) J_3\left(\sqrt{\lambda_{3n}}\, r\right)}$$

Using the orthogonality of $J_3\left(\sqrt{\lambda_{mn}}\, r\right)$ with weight r:

$$\boxed{A_n c\sqrt{\lambda_{3n}} = \frac{\int_0^a \alpha(r) J_3\left(\sqrt{\lambda_{3n}}\, r\right) r\, dr}{\int_0^a J_3\left(\sqrt{\lambda_{3n}}\, r\right)^2 r\, dr}}$$