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Week 12 7.8.10
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In this exercise use the result of Exercise 7.8.7 $\left[\frac{d^2y}{dz^2}+y\left(1+\frac{1}{4}z^{-2}-m^2z^{-2}\right)=0\right]$ **in order to improve on (7.8.3):**

a. Substitute $y=e^{iz}w(z)$ **and show that**

$$\frac{d^2w}{dz^2}+2i\frac{dw}{dz}+\frac{\gamma}{z^2}w=0, \text{ where } \gamma=\frac{1}{4}-m^2$$

b. Substitute $w=\sum_{n=0}^{\infty}\beta_n z^{-n}$ **. Determine the first few terms** β_n **(assuming that** $\beta_0=1$**).**

c. Use part (b) to obtain an improved asymptotic solution of Bessel's differential equation. For real solutions, take real and imaginary parts.

d. Find a recurrence formula for β_n**. Show that the series diverges. (Nonetheless, a finite series is very useful.)**

a.

$$y=e^{iz}w(z), \qquad \frac{dy}{dz}=ie^{iz}w(z)+e^{iz}w'(z),$$

$$\frac{d^2y}{dz^2}=\left(i^2e^{iz}w(z)-ie^{iz}w'(z)\right)+\left(ie^{iz}w'(z)+e^{iz}w''(z)\right)=-e^{iz}w(z)+2ie^{iz}w'(z)+e^{iz}w''(z)$$

$$\frac{d^2y}{dz^2}+y\left(1+\frac{1}{4}z^{-2}-m^2z^{-2}\right)=\left[-e^{iz}w(z)+2ie^{iz}w'(z)+e^{iz}w''(z)\right]+e^{iz}w(z)\left(1+\frac{1}{4}z^{-2}-m^2z^{-2}\right)=0$$

$$-\cancel{e^{iz}}\cancel{w(z)}+2i\cancel{e^{iz}}w'(z)+\cancel{e^{iz}}w''(z)+\cancel{e^{iz}}\cancel{w(z)}+\left(\frac{1}{4}+m^2\right)z^{-2}\cancel{e^{iz}}w(z)=0$$

$$\frac{d^2w}{dz^2}+2i\frac{dw}{dz}+\frac{\left(\frac{1}{4}+m^2\right)}{z^2}w=\frac{d^2w}{dz^2}+2i\frac{dw}{dz}+\frac{\gamma}{z^2}w=0$$

b. $w(z)=\sum_{n=0}^{\infty}\beta_n z^{-n}$, $\beta_0=1$

$$w'(z)=\sum_{n=1}^{\infty}-n\beta_n z^{-n-1}=\sum_{n=0}^{\infty}-(n+1)\beta_{n+1}z^{-n-1+1}=\sum_{n=0}^{\infty}(-n-1)\beta_{n+1}z^{-n}$$

$$w''(z)=\sum_{n=2}^{\infty}-n(-n-1)\beta_n z^{-n-2}=\sum_{n=0}^{\infty}-(n+2)(-(n+2)-1)\beta_{n+2}z^{-n-2+2}=\sum_{n=0}^{\infty}(-n-2)(-n-3)\beta_{n+2}z^{-n}$$

When n=0, the equation from part (a), after some algebra, yields: $\beta_2=\frac{1}{6}\left(-\frac{\gamma}{z^2}+2i\beta_1\right)$.

When n=1, the equation yields: $\beta_3=\frac{1}{12}\left(4i\beta_2-\frac{\lambda}{z^2}\beta_1\right)=-\frac{i}{18}\frac{\gamma}{z^2}+\left(\frac{i}{6}-\frac{1}{12}\frac{\gamma}{z^2}\right)\beta_1$.

When n=2, the equation yields: $\beta_4 = \frac{1}{20}\left(-\frac{\gamma}{z^2}\beta_2 + 6i\beta_3\right) = \frac{1}{120}\left(\frac{\gamma}{z^2}\right)^2 + \frac{1}{60}\frac{\gamma}{z^2} + \left(\frac{2i}{120} - \frac{1}{20} - \frac{i}{40}\frac{\gamma}{z^2}\right)\beta_1$.

Similarly, for all β_n with $n>1$, can be represented in terms of β_1.

c.

d.