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Assignment 7 10.6.10

We are asked to solve $\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$ subject to the initial condition $u(x,y,0) = f(x,y)$.

We begin by redefining the PDE as a spatial double Fourier transform:

$$\frac{\partial \bar{U}}{\partial t} = -(k_1\omega_1^2 + k_2\omega_2^2)\bar{U} \quad (1)$$

$$\bar{U}(\boldsymbol{\omega}, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)e^{i\boldsymbol{\omega}\cdot\boldsymbol{r}}dxdy \quad (2)$$

where $\boldsymbol{\omega}$ and $\boldsymbol{r}$ are the wave number vector and position vector, respectively. Additionally we will define the "**k**" vector:

$$\boldsymbol{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} \quad (3)$$

$$\boldsymbol{r} = x\hat{\imath} + y\hat{\jmath} \quad (4)$$

$$\boldsymbol{k} = k_1\hat{\imath} + k_2\hat{\jmath} \quad (5)$$

The "elementary" solution, according to Haberman, can then be presented as:

$$\bar{U}(\boldsymbol{\omega}, t) = A(\boldsymbol{\omega})e^{-\boldsymbol{k}\cdot\boldsymbol{\omega}^2 t} \quad (6)$$

$$A(\boldsymbol{\omega}) = \bar{U}(\boldsymbol{\omega}, 0) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i\boldsymbol{\omega}\cdot\boldsymbol{r}}dxdy \quad (7)$$

Thus, we can find a generic solution to this heat equation:

$$u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega)e^{-\boldsymbol{k}\cdot\boldsymbol{\omega}^2 t}e^{-i\boldsymbol{\omega}\cdot\boldsymbol{r}}d\omega_1\, d\omega_2 \quad (8)$$

The convolution theorem is then applied, using the fact that $H(\boldsymbol{\omega}) = F(\boldsymbol{\omega})G(\boldsymbol{\omega})$.

$$h(\boldsymbol{r}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{r}_0)g(\boldsymbol{r} - \boldsymbol{r}_0)dx_0\, dy_0 \quad (9)$$

With the knowledge that our solution is the product of $A(\boldsymbol{\omega})$ and $e^{-\boldsymbol{k}\cdot\boldsymbol{\omega}^2 t}$, we can present a solution with the latter equation's double Fourier transform.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\boldsymbol{k}\cdot\boldsymbol{\omega}^2 t}e^{-i\boldsymbol{\omega}\cdot\boldsymbol{r}}d\omega_1\, d\omega_2 \quad (10)$$

$$= \int_{-\infty}^{\infty} e^{-k_1\omega_1^2 t}e^{-i\omega_1 x}d\omega_1 \int_{-\infty}^{\infty} e^{-k_2\omega_2^2 t}e^{-i\omega_2 y}\, d\omega_2 \quad (11)$$

$$= \sqrt{\frac{\pi}{k_1 t}}e^{-x^2/4k_1 t}\sqrt{\frac{\pi}{k_2 t}}e^{-y^2/4k_2 t} \quad (12)$$

Equation (12) corresponds with our Green's function, $g(\boldsymbol{r})$. Thus, because $h(\boldsymbol{r})$ corresponds to our solution, $u(\boldsymbol{r})$, we can write:

$$h(x, y, t; x_0, y_0, 0) = u(x, y, t) =$$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_0, y_0) \frac{\pi}{t\sqrt{k_1 k_2}} \exp\left[-\frac{(x-x_0)^2}{4tk_1} - \frac{(y-y_0)^2}{4tk_2}\right] dx_0\, dy_0 \qquad (13)$$

Thus, our final solution is:

$$\boldsymbol{u(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_0, y_0) \frac{1}{4t\pi\sqrt{k_1 k_2}} \exp\left[-\frac{(x-x_0)^2}{4tk_1} - \frac{(y-y_0)^2}{4tk_2}\right] dx_0\, dy_0} \qquad \textbf{(14)}$$