Matthew Gidden mgidden http://yojox.com/mgidden/math412/7.10.3c.pdf
Assignment 12 7.10.3(c)

We are asked to solve the heat equation in a sphere with a Neumann boundary condition. The sphere has a radius a. We begin by stating the problem and conditions and then start solving the problem using separation of variables.

$$\frac{\partial u}{\partial t} = k\nabla^2 u \qquad (1)$$

$$\frac{\partial u}{\partial r}(a, \theta, \varphi, t) = 0 \qquad (2)$$

$$u(r, \theta, \varphi, t) = F(r, \varphi)\sin(3\theta) \qquad (3)$$

$$\frac{T'}{kT} = \frac{\nabla^2 u}{\mathrm{TR}\boldsymbol{\varphi\theta}} = -\omega \qquad (4)$$

We can see that our function of theta must be a sin series by (3). Additionally, our time dependence is exponential as is always the case for the heat equation.

$$T = \exp(-\omega k t) \qquad (5)$$

$$\boldsymbol{\theta} = \sin(m\theta) \qquad (6)$$

Using the Laplacian in spherical coordinates, we are left to solve two equations:

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + (\omega r^2 - \mu)R = 0 \qquad (7)$$

$$\frac{d}{d\varphi}\left(\sin(\varphi)\frac{d\boldsymbol{\varphi}}{d\varphi}\right) + \left(\mu\sin(\varphi) - \frac{m^2}{\sin(\varphi)}\right)\boldsymbol{\varphi} = 0 \qquad (8)$$

By making a change of variables, $\mathrm{x} = \cos(\varphi)$, (8)'s solution can be seen as the famous Legendre functions for $m > 0$ with eigenvalues $\mu = n(n+1)$:

$$\boldsymbol{\varphi} = \mathrm{P_n^m}(\cos(\varphi)) \qquad (9)$$

Given mu, we can continue to solve (7). Recognizing that (7) is a spherical Bessel's function, we can see that the solution takes the form:

$$r^{-1/2}\mathrm{J}_{\mathrm{n}+\frac{1}{2}}(\omega \mathrm{r}) \qquad (10)$$

We find reciprocal relations for omega using (2):

$$R' = r^{-\frac{1}{2}}\frac{\left(\mathrm{J}_{\mathrm{n}-\frac{1}{2}}(\omega\mathrm{r}) - \mathrm{J}_{\mathrm{n}+\frac{3}{2}}(\omega\mathrm{r})\right)}{2} - \frac{r^{-\frac{3}{2}}}{2}\mathrm{J}_{\mathrm{n}+\frac{1}{2}}(\omega\mathrm{r}) = 0 \qquad (11)$$

$$\mathrm{r}\left(\mathrm{J}_{\mathrm{n}-\frac{1}{2}}(\omega\mathrm{r}) - \mathrm{J}_{\mathrm{n}+\frac{3}{2}}(\omega\mathrm{r})\right) = \mathrm{J}_{\mathrm{n}+\frac{1}{2}}(\omega\mathrm{r}) \qquad (12)$$

We can then see the equation for u:

$$\boldsymbol{u(r,\theta,\varphi,t)} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \mathbf{C_{nm}} \mathbf{sin}\,(\boldsymbol{m\theta})\mathbf{P_n^m}\,(\mathbf{cos}\,(\boldsymbol{\varphi}))\boldsymbol{r^{-1/2}}\mathbf{J_{n+\frac{1}{2}}}(\boldsymbol{\omega}\mathbf{r})\mathbf{exp}\,(-\boldsymbol{\omega k t}) \quad \textbf{(13)}$$

We then fully solve our problem using (3):

$$F(r,\varphi) = \frac{\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} \mathrm{C_{mn}} \sin\,(m\theta)\mathrm{P_n^m}\,(\cos\,(\varphi))r^{-1/2}\mathrm{J_{n+\frac{1}{2}}}(\omega \mathrm{r})}{\sin\,(3\theta)} \quad (14)$$

$$\mathbf{C_{mn}} = \frac{\int\int \boldsymbol{F(r,\varphi)}\,\mathbf{si\,n}(\boldsymbol{m\theta})\mathbf{P_n^m}(\mathbf{co\,s}(\boldsymbol{\varphi}))\boldsymbol{r}^{-\frac{1}{2}}\mathbf{J_{n+\frac{1}{2}}}(\boldsymbol{\omega}\mathbf{r})\mathbf{sin}(\mathbf{3}\boldsymbol{\theta})}{\int\int\left[\mathbf{sin}\,(\boldsymbol{m\theta})\mathbf{P_n^m}\,(\mathbf{cos}\,(\boldsymbol{\varphi}))\boldsymbol{r^{-1/2}}\mathbf{J_{n+\frac{1}{2}}}(\boldsymbol{\omega}\mathbf{r})\right]^2} \quad \textbf{(14)}$$