(b) We are asked to solve for u(x,y,t) on (0<x<L), (0<y<H), (0<t<∞) with the PDE, BCs, and IC as follows:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (1)$$

$$u(0,y,t) = 0, \quad \frac{\partial u}{\partial x}(L,y,t) = 0 \qquad (2)$$

$$\frac{\partial u}{\partial y}(x,0,t) = 0, \quad \frac{\partial u}{\partial y}(x,H,t) = 0 \qquad (3)$$

$$u(x,y,0) = f(x,y) \qquad (4)$$

To do so we must first separate variables.

$$u(x,y,t) = X(x)Y(y)T(t) \qquad (5)$$

$$T'XY = k(X''YT + Y''XT) = 0 \qquad (6)$$

$$\frac{T'}{Tk} = \frac{X''}{X} + \frac{Y''}{Y} = -\varphi \qquad (7)$$

$$\frac{X''}{X} = -\mu, \quad \frac{Y''}{Y} = -\varphi + \mu \qquad (8)$$

We can then apply the boundary conditions to obtain:

$$X(x) = Asin(\sqrt{\mu}x) + Bcos(\sqrt{\mu}x) \qquad (9)$$

$$X'(x) = \sqrt{\mu}Acos(\sqrt{\mu}x) - \sqrt{\mu}Bsin(\sqrt{\mu}x) \qquad (10)$$

$$X(0) = 0 = B \qquad (11)$$

$$X'(L) = 0 = \sqrt{\mu}Acos(\sqrt{\mu}L) \qquad (12)$$

$$\sqrt{\mu} = \frac{\left(m-\frac{1}{2}\right)\pi}{L} \qquad (13)$$

We can find Y(y) in a similar fashion.

$$Y(y) = Csin\left(\sqrt{\varphi-\mu}y\right) + Dcos(\sqrt{\varphi-\mu}y) \qquad (14)$$

$$Y'(y) = \sqrt{\varphi-\mu}Ccos\left(\sqrt{\varphi-\mu}y\right) - \sqrt{\varphi-\mu}Dsin(\sqrt{\varphi-\mu}y) \qquad (15)$$

$$Y'(0) = 0 = \sqrt{\varphi-\mu}C \qquad (16)$$

$$Y'(H) = 0 = \sqrt{\varphi - \mu} D sin(\sqrt{\varphi - \mu} H) \qquad (17)$$

$$\sqrt{\varphi - \mu} = \frac{n\pi}{H} \qquad (18)$$

$$\boldsymbol{\varphi = \left(\frac{n\pi}{H}\right)^2 + \left(\frac{\left(m-\frac{1}{2}\right)\pi}{L}\right)^2} \qquad \textbf{(19)}$$

Additionally, we can solve for T(t), completing all of the equations necessary for u(x,y,t).

$$T(t) = e^{-\varphi k t} \qquad (20)$$

With φ, our eigenvalues, we can write:

$$\boldsymbol{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \mathrm{c_{mn}} \sin\left(\frac{\left(m-\frac{1}{2}\right)\pi}{L} x\right) \cos\left(\frac{n\pi}{H} y\right) e^{-\varphi k t}} \qquad \textbf{(21)}$$

The initial condition must then be used in order to determine $\mathrm{c_{mn}}$.

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \mathrm{c_{mn}} \sin\left(\frac{\left(m-\frac{1}{2}\right)\pi}{L} x\right) \cos\left(\frac{n\pi}{H} y\right) \qquad (22)$$

The Fourier formulas are then applied twice:

$$\sum_{n=0}^{\infty} \mathrm{c_{mn}} \cos\left(\frac{n\pi}{H} y\right) = \frac{2}{L} \int_0^L f(x,y) \sin\left(\frac{\left(m-\frac{1}{2}\right)\pi}{L} x\right) dx \qquad (23)$$

$$\boldsymbol{\mathrm{c_{mn}} = \frac{2}{L}\frac{2}{H} \int_0^H dy \int_0^L dx \left[f(x,y) \sin\left(\frac{\left(m-\frac{1}{2}\right)\pi}{L} x\right) \cos\left(\frac{n\pi}{H} y\right)\right]} \qquad \textbf{(24)}$$

$$\boldsymbol{\mathrm{c_{m0}} = \frac{2}{LH} \int_0^H dy \int_0^L dx \left[f(x,y) \sin\left(\frac{\left(m-\frac{1}{2}\right)\pi}{L} x\right)\right]} \qquad \textbf{(25)}$$