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(a) We are asked to solve for the heat equation inside a quarter-circular cylinder ($0 < \theta < \frac{\pi}{2}$, radius a, and height H) under the following conditions:

$$\frac{\partial u}{\partial t} = k\Delta u \qquad (1)$$

$$u(r,\theta,0) = 0, \quad u(r,\theta,H) = 0 \qquad (2)$$

$$u(r,0,z) = 0, \quad u\left(r,\theta,\frac{\pi}{2}\right) = 0 \qquad (3)$$

$$u(a,\theta,z) = 0, \quad u(r,\theta,z,0) = f(r,\theta,z) \qquad (4)$$

To do so we must first separate variables.

$$u(r,\theta,z,t) = X(x)T(t) \qquad (5)$$

$$XT' = k\Delta XT \qquad (6)$$

$$\frac{T\prime}{Tk} = -\varphi \qquad (7)$$

$$T = \exp\left(-\varphi kt\right) \qquad (8)$$

We can then further separate X to obtain:

$$X(x) = R(r)\vartheta(\theta)Z(z) \qquad (9)$$

$$\Delta X = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\vartheta}\frac{d^2\vartheta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + \varphi = 0 \qquad (10)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} + \varphi = \omega \qquad (11)$$

$$Z'' = (\omega - \varphi)Z \qquad (12)$$

Using our Z-dependent boundary conditions, we can arrive at:

$$Z = \sin\left(\sqrt{\omega - \varphi} z\right), \omega - \varphi = \left(\frac{p\pi}{H}\right)^2 \qquad (13)$$

Continuing with our separated solution:

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\vartheta}\frac{d^2\vartheta}{d\theta^2} + \omega = 0 = -m \qquad (14)$$

We then multiply by r^2 and solve for each equation separately:

$$\frac{d^2\vartheta}{d\theta^2} = -m\vartheta \qquad (15)$$

Again, implementing the relevant boundary conditions, we find that:

$$\vartheta = \sin(2n\theta) \qquad (16)$$

Finally, R is solved for:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \omega r^2 = -4n^2 \qquad (17)$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\omega r^2 - 4n^2)R = 0 \qquad (18)$$

We can see that this is a form of the Bessel Equation, thus we find that:

$$R = CJ_{2n}(\sqrt{\omega}r) \qquad (19)$$

Our total solution is then:

$$\boldsymbol{u(r,\theta,z,t) = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{pnm}J_{2n}(\sqrt{\omega_{mn}}r)\sin\left(\frac{p\pi}{H}z\right)\sin(2n\theta)\exp(-\varphi_{pmn}kt)} \qquad \textbf{(20)}$$

Using the initial condition:

$$\boldsymbol{C_{pnm} = \frac{2}{\pi H}\frac{\int_0^L\int_0^a\int_0^{\frac{\pi}{2}} f(r,\theta,z)\sin\left(\frac{p\pi}{H}z\right)\sin(2n\theta)J_{2n}(\sqrt{\omega_{mn}}r)\,r\,dr\,dz\,d\theta}{\int_0^a J_{2n}(\sqrt{\omega_{mn}}r)^2\,r\,dr}} \qquad \textbf{(21)}$$

As t approaches infinity, we can apply common sense to know that the object will cool back to ambient temperature (u goes to 0). This can also be shown because the t related eigenvalues will be positive, thus the exponent will approach 0 as t goes to infinity.