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Assignment 8 Problem 5

We are asked to solve the familiar heat equation by separating variables or using an equivalent transforming technique. The given equation and initial condition are as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad for \quad x \in (-\infty, \infty) \tag{1}$$

$$u(0,x) = f(x) \tag{2}$$

The solution is fairly straightforward by using spatial Fourier transforms on the real line. We will first transform u(t,x):

$$U(t,k) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{ikx} u(t,x) dx \tag{3}$$

The Fourier transform is then inserted into the original PDE:

$$\frac{\partial U}{\partial t} = -k^2 U \tag{4}$$

$$U(t,k) = c(k) e^{-k^2 t} \tag{5}$$

The transformed initial condition is then applied:

$$U(0,k) = F(k) \tag{6}$$

$$F(k) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx \tag{7}$$

We then apply the inverse transform on U:

$$u(t,x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} U(t,k) dk \tag{8}$$

$$u(t,x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} dk e^{-ikx} \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} d\tilde{x} e^{ik\tilde{x}} f(\tilde{x})\, e^{-k^2 t} \tag{9}$$

$$\boldsymbol{u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\tilde{x} e^{-ikx} e^{-k^2 t} e^{ik\tilde{x}} f(\tilde{x})} \tag{10}$$

The solution can be separated into the familiar definition of a Greens function multiplied by the initial condition:

$$u(t,x) = \int_{-\infty}^{\infty} H(x-z) f(z) dz \tag{11}$$

In our problem, we can separate the solution, finding H:

$$\boldsymbol{u(t,x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x-z)} e^{-k^2 t}\, dk\right] f(z) dz} \tag{12}$$

$$\boldsymbol{\frac{1}{2\pi} H(x-z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(x-z)} e^{-k^2 t} dk = \frac{1}{\sqrt{4\pi t}} e^{-(x-z)^2/4t}} \tag{13}$$