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Math 412, Problem 12.5.3
Available online at www.people.tamu.edu/~granthelmreich/exercise12.5.3

12.5.3 a) The function to be analyzed is defined by the following equations,

$$\frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2} \qquad 0 < x < 10 \qquad \text{(Equation 1)}$$

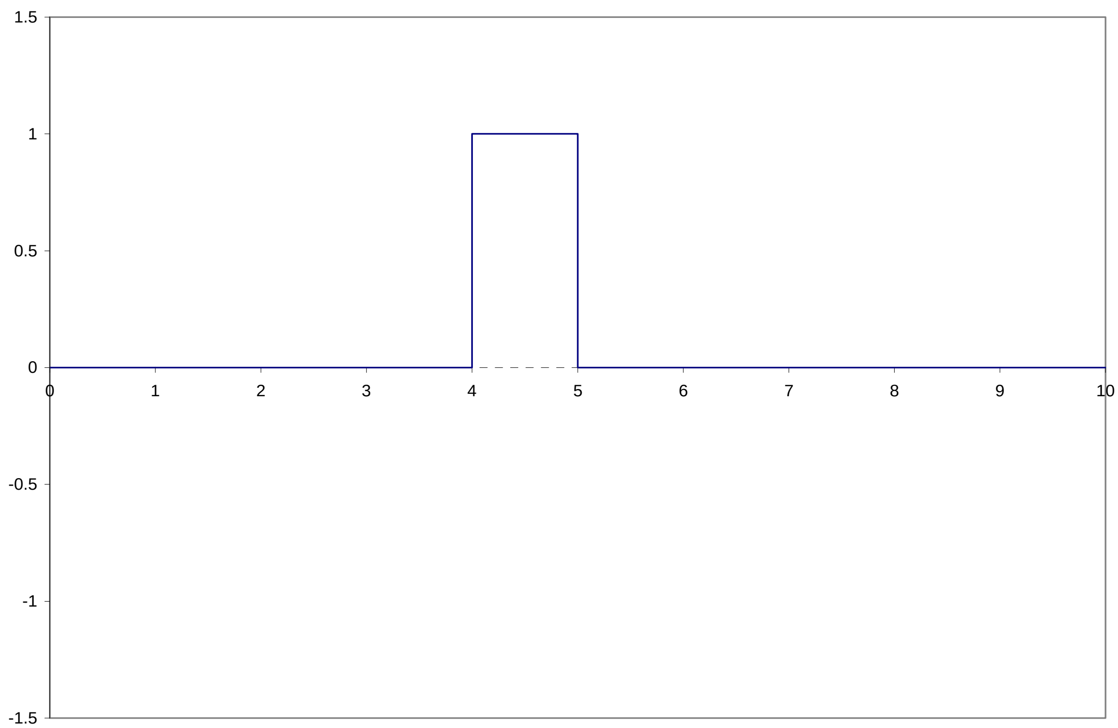
$$u(x,0) = f(x) = \left\{ \frac{1 \rightarrow 4 < x < 5}{0 \rightarrow otherwise} \right\} \qquad \text{(Equation 2)}$$

$$u(0,t) = 0 \qquad \text{(Equation 3)}$$

$$\frac{\delta u}{\delta t}(x,0) = g(x) = 0 \qquad \text{(Equation 4)}$$

$$\frac{\delta u}{\delta x}(L,t) = 0 \qquad \text{(Equation 5)}$$

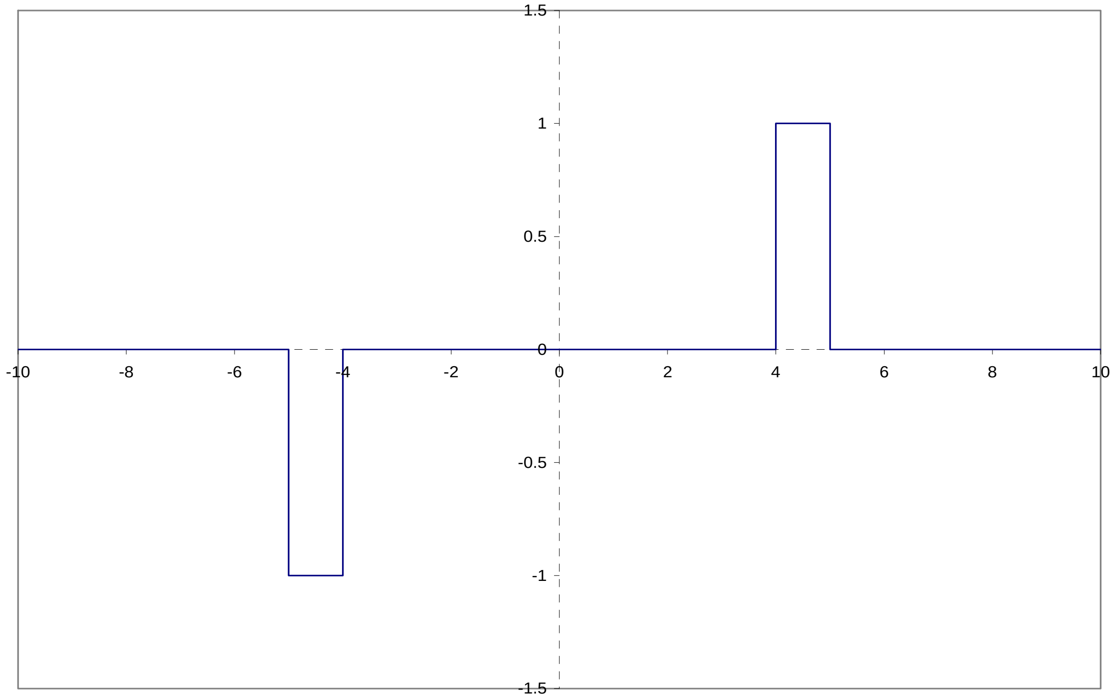
Based on these equations and the method of characteristics, the basic solution can be sketched as follows.



(Figure 1)

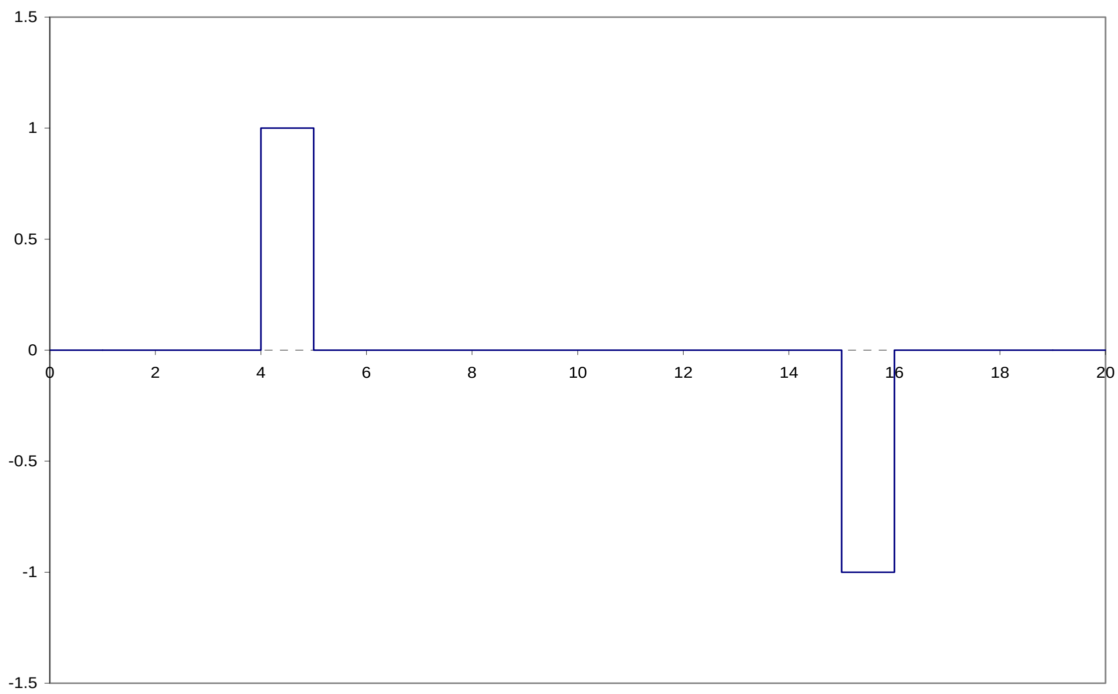
Figure 1 shows a single characteristic of *u*(x,t) obtained by setting t = 0. Other characteristics may be obtained by setting different values of t and plotting *f*(x-ct) and *f*(x+ct).

c) If the initial function $f(x)$ is given an odd extension across the left boundary, a result suggestive of periodicity is obtained.



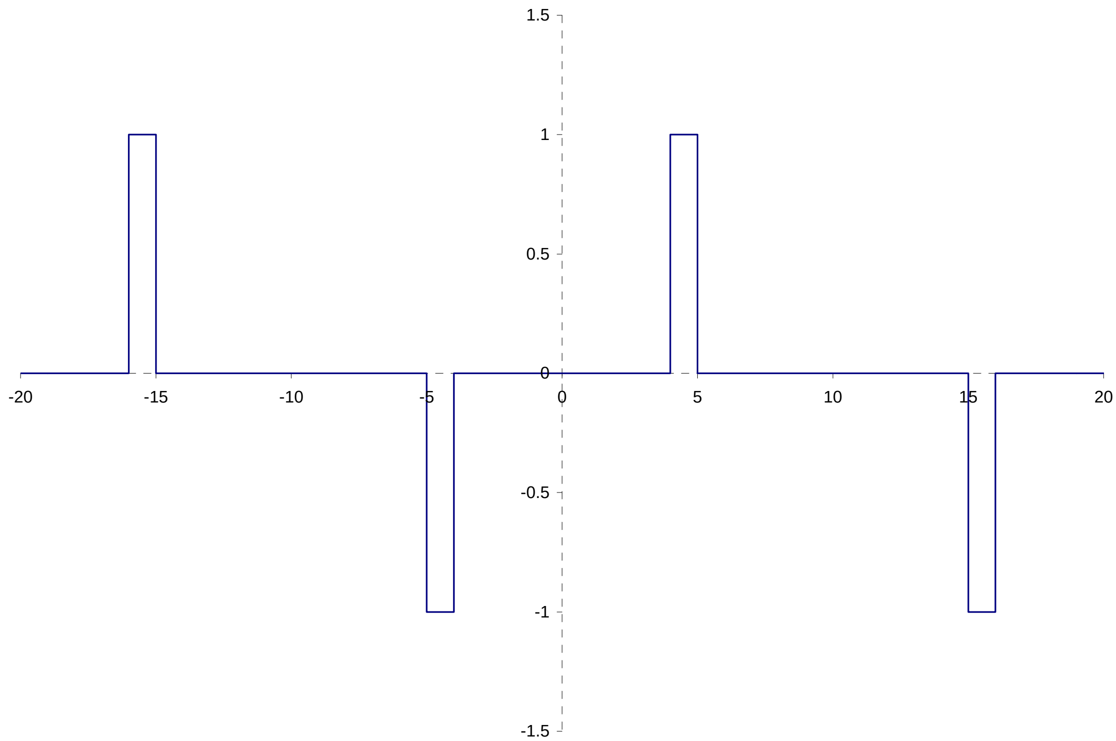
(Figure 2)

Extending $f(x)$ across the right boundary in a similar fashion results in the following.



(Figure 3)

Both Figure 2 and Figure 3 seem to suggest periodicity, however, a period greater than the initial 10 units must be present. Combining left and right extensions and adding one more to the left clearly shows a period of 20 units as seen below.



(Figure 4)

These extensions could theoretically be continued infinitely, and thus a new equation f_{ext}(x) may be defined.

$$f_{ext}(x) = \left\{ \begin{array}{c} 1 \rightarrow 4 < x - 20n < 5 \\ \hline -1 \rightarrow 15 < x - 20n < 16 \\ \hline 0 \rightarrow otherwise \end{array} \right\} \quad -\infty < n < \infty \qquad \text{(Equation 6)}$$

In this extension, n is defined as any integer. The inclusion of a factor of 20n gives f_{ext} the desired periodicity. The validity of the extended function can be tested be setting n=0 or n=-1 and comparing to Figure 4.

Finally, f_{ext} must be modified to include the second variable within its expression by adding or subtracting a factor of ct.

$$f_{ext}(x-ct) = \left\{ \begin{array}{c} 1 \to 4 < (x-ct) - 20n < 5 \\ \hline -1 \to 15 < (x-ct) - 20n < 16 \\ \hline 0 \to otherwise \end{array} \right\} -\infty < n < \infty \qquad \text{(Equation 7)}$$

$$f_{ext}(x+ct) = \left\{ \begin{array}{c} 1 \to 4 < (x+ct) - 20n < 5 \\ \hline -1 \to 15 < (x+ct) - 20n < 16 \\ \hline 0 \to otherwise \end{array} \right\} -\infty < n < \infty \qquad \text{(Equation 8)}$$

Now that f_{ext} has been defined for both variables, u(x,t) can be found by d'Alembert's solution.

$$u(x,t) = \frac{f_{ext}(x-ct) + f_{ext}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x} \qquad \text{(Equation 9)}$$

However, Equation 4 defines g(x) to be zero, so that d'Alembert's solution in this case reduces to,

$$u(x,t) = \frac{1}{2}[f_{ext}(x-ct) + f_{ext}(x+ct)] \qquad \text{(Equation 10)}$$

Given the definitions for f_{ext} provide by Equation 7-8, this comprises a final solution of u(x,t) extended over the infinite domain.