Grant Helmreich NetID: granthelmreich Math 412 Homework 11/26/07

PDF available online at: people.tamu.edu/~granthelmreich/exercise2.4.1

2.4.1 (a) Solve the equation,

$$\frac{\delta u}{\delta t} = k\frac{\delta^2 u}{\delta x^2}, \quad 0 < x < L, \quad 0 < t \tag{1}$$

Given boundary conditions,

$$\frac{\delta u}{\delta x}(0,t) = 0 \tag{2}$$

$$\frac{\delta u}{\delta x}(L,t) = 0 \tag{3}$$

And initial conditions,

$$u(x,0) = f(x) = \begin{Bmatrix} 0....x < L/2 \\ 1...x > L/2 \end{Bmatrix} \tag{4}$$

By separation of variables, we assume based on experience that the solution if of form,

$$u(x,t) = \phi(x)G(t) \tag{5}$$

Now by taking partial differentials with respect to *t* and *x* and substitution into equation 1,

$$\frac{\phi''(x)}{\phi(x)} = \frac{G'(t)}{kG(t)} \tag{6}$$

Equation 6 then directly implies,

$$\frac{\delta G}{\delta t} = -\lambda k G \tag{7}$$

$$\frac{\delta^2 \phi}{\delta x^2} = -\lambda \phi \tag{8}$$

Equation 7 is a simple ODE with a standard solution,

$$G(t) = ce^{-\lambda k t} \tag{9}$$

By recognizing that this problem uses Neumann boundary conditions, experience shows that equation 8 implies,

$$\phi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) = c_1 \cos\left(\frac{n\pi x}{L}\right) \tag{10}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \tag{11}$$

Substitution into the equation 5 yields,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt} \tag{12}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \tag{13}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{14}$$

Substitution of equation 4 for *f(x)* yields,

$$A_0 = \frac{1}{L}\left(L - \frac{L}{2}\right) = \frac{1}{2} \tag{15}$$

$$A_n = \frac{2}{L}\left(-\sin\left(\frac{n\pi}{2}\right)\frac{L}{n\pi}\right) = -\sin\left(\frac{n\pi}{2}\right)\frac{2}{n\pi} \tag{16}$$

Combining equations 12, 15, and 16 yields the final solution for *u(x,t)*,

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[-\sin\left(\frac{n\pi}{2}\right)\frac{2}{n\pi}\right]\cos\left(\frac{n\pi x}{L}\right)e^{-(n\pi/L)^2 kt} \tag{17}$$

2.4.1 (b) Solve the equation,

$$\frac{\delta u}{\delta t} = k\frac{\delta^2 u}{\delta x^2}, \quad 0 < x < L, \quad 0 < t \tag{1}$$

Given boundary conditions,

$$\frac{\delta u}{\delta x}(0,t) = 0 \tag{2}$$

$$\frac{\delta u}{\delta x}(L,t) = 0 \tag{3}$$

And initial conditions,

$$u(x,0) = f(x) = 6 + 4\cos\left(\frac{3\pi x}{L}\right) \tag{4}$$

By separation of variables, we assume based on experience that the solution if of form,

$$u(x,t) = \phi(x)G(t) \tag{5}$$

Now by taking partial differentials with respect to *t* and *x* and substitution into equation 1,

$$\frac{\phi''(x)}{\phi(x)} = \frac{G'(t)}{kG(t)} \tag{6}$$

Equation 6 then directly implies,

$$\frac{\delta G}{\delta t} = -\lambda kG \tag{7}$$

$$\frac{\delta^2 \phi}{\delta x^2} = -\lambda \phi \tag{8}$$

Equation 7 is a simple ODE with a standard solution,

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By recognizing that this problem uses Neumann boundary conditions, experience shows that equation 8 implies,

$$\phi(x) = c_1 \cos\left(\sqrt{\lambda}x\right) = c_1 \cos\left(\frac{n\pi x}{L}\right) \tag{10}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \tag{11}$$

Substitution into the equation 5 yields,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt} \tag{12}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \tag{13}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{14}$$

Substitution of equation 4 for *f(x)* yields,

$$A_0 = \frac{1}{L}\left[\left(6L + \frac{4L}{3\pi}\sin(3\pi)\right) - \left(0 + \frac{4L}{3\pi}\sin(0)\right)\right] = 6 \tag{15}$$

$$A_n = \frac{2}{L}\int_0^L \left[6 + 4\cos\left(\frac{3\pi x}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right)dx \tag{16}$$

At this point, orthogonality conditions show that A_n is zero for all values except $n = 3$. Solving at this value of n yields,

$$A_3 = \frac{2}{L}\int_0^L \left[6\cos\left(\frac{3\pi x}{L}\right) + 4\cos^2\left(\frac{3\pi x}{L}\right)\right]dx = \frac{2}{L}\left|\frac{2L}{\pi}\sin\left(\frac{3\pi x}{L}\right) + \frac{4x}{2} + \frac{L}{3\pi}\sin\left(\frac{6\pi x}{L}\right)\right|_0^L \tag{17}$$

$$A_3 = \frac{2}{L}\left[(0 + 2L + 0) - (0 + 0 + 0)\right] = 4 \tag{18}$$

Combining equations 12, 15, and 18 yields the final solution for *u(x,t)*,

$$u(x,t) = 6 + 4\left[\cos\left(\frac{3\pi x}{L}\right)e^{-(3\pi/L)^2 kt}\right] \tag{19}$$