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PDF available online at: people.tamu.edu/~granthelmreich/exercise2.5.3

a) Equation 1 defines Laplace's Equation in polar coordinates,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{1}$$

With the given boundary condition,

$$u(a,\theta) = \ln 2 + 4\cos(3\theta) \tag{2}$$

By periodicity we may state that,

$$u(r,-\pi) = u(r,\pi) \tag{3}$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \tag{4}$$

We may also obtain a second boundary condition by requiring boundedness at infinity,

$$\left| u(\infty,\theta) \right| < \infty \tag{5}$$

Assume that solutions will be of the form,

$$u(r,\theta) = \phi(\theta)G(r) \tag{6}$$

By separation of variables and application of Equation 1,

$$\frac{r}{G}\frac{dG}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda \tag{7}$$

Application of periodicity conditions to the angular portion of Equation 7 yields,

$$\lambda = \left(\frac{n\pi}{L} \right)^2 = n^2 \tag{8}$$

Which has eigenfunctions of $\sin(n\pi)$ and $\cos(n\pi)$, including the case n = 0.

The radial dependent problem may then be written as,

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0 \tag{9}$$

The general solution to Equation 9 is,

$$G = c_1 r^n + c_2 r^{-n} \tag{10}$$

However, by boundedness at infinity as stated in Equation 5, c_1 must be zero. Thus,

$$G = c_2 r^{-n} \qquad n \geq 0 \tag{11}$$

Multiplication of the separated equations and summation over the index yields,

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta) \qquad (12)$$

$$a \le r \le \infty \qquad\qquad -\pi < \theta \le \pi$$

Finally, introduction of the nonhomogeneous condition stated in Equation 2 yields,

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln 2 + 4\cos(3\theta) d\theta \qquad (13)$$

$$A_n a^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} (\ln 2 + 4\cos(3\theta)) \cos(n\theta) d\theta \qquad (14)$$

$$B_n a^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} (\ln 2 + 4\cos(3\theta)) \sin(n\theta) d\theta \qquad (15)$$

These integrals may be evaluated using orthogonality to yield,

$$A_0 = \ln 2 \qquad (16)$$

$$A_3 a^{-3} = 4 \qquad (17)$$

All other A_n and B_n are zero.

(b) The following solution is identical to part (a) until the final steps where the nonhomogenous boundary condition is applied.

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$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (1)$$

With the given boundary condition,

$$u(a,\theta) = f(\theta) \qquad (2)$$

By periodicity we may state that,

$$u(r,-\pi) = u(r,\pi) \qquad (3)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \qquad (4)$$

We may also obtain a second boundary condition by requiring boundedness at infinity,

$$|u(\infty,\theta)| < \infty \qquad (5)$$

Assume that solutions will be of the form,

$$u(r,\theta) = \phi(\theta)G(r) \qquad (6)$$

By separation of variables and application of Equation 1,

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The radial dependent problem may then be written as,

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The general solution to Equation 9 is,

$$G = c_1 r^n + c_2 r^{-n} \qquad (10)$$

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$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta) \qquad a \le r \le \infty \qquad -\pi < \theta \le \pi \tag{12}$$

Finally, introduction of the nonhomogeneous condition stated in Equation 2 yields,

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