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Available online at http://people.tamu.edu/~granthelmreich/exercise5.4.1

a) The Rayleigh quotient for this problem may be used to define the eigenvalues in terms of their respective eigenfunctions and the physical coefficients of the system.

$$\lambda = \frac{-K_0\phi \left.{}^{d\phi}\!/\!_{dx}\right|_0^L + \int_0^L K_0\left({}^{d\phi}\!/\!_{dx}\right)^2 - \alpha\phi^2 dx}{\int_0^L \phi^2 c\rho dx} \tag{1}$$

This expression may be simplified by application of the boundary conditions to the following form:

$$\lambda = \frac{\int_0^L K_0\left({}^{d\phi}\!/\!_{dx}\right)^2 - \alpha\phi^2 dx}{\int_0^L \phi^2 c\rho dx} \tag{2}$$

In the special case when $\alpha < 0$, each term on the right side of the equation must be positive, leading to the conclusion that for all $\alpha < 0$, the eigenvalues must be positive.

b) By separation of variables following the Sturm-Liouville method,

$$u(x,t) = \phi(x)h(t) \tag{3}$$

$$\frac{dh}{dt} = -\lambda h \tag{4}$$

$$\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \lambda c\rho\phi + \alpha\phi = 0 \tag{5}$$

The solution to the time-dependent relation is known by experience to be,

$$h(t) = ce^{-\lambda_n t} \tag{6}$$

By the assumption that eigenfunctions are known for given eigenvalues, the complete function may be defined as,

$$u(x,t) = \sum_{n=1}^{\infty} a_n\phi_n(x)e^{-\lambda_n t} \tag{7}$$

Addition of the initial conditions and orthogonality relations defines the coefficients of this function in the following manner,

$$a_n = \frac{\int_0^L f(x)\phi_n(x)c(x)\rho(x)dx}{\int_0^L \phi_n^2(x)c(x)\rho(x)dx} \tag{8}$$

c) In general, as t approaches infinity the function may be approximated by the first term,

$$u(x,t) \approx a_1\phi_1(x)e^{-\lambda_1 t} \tag{9}$$

In the special case in which all λ are positive (see part a), Equation 9 will approach zero as t approaches infinity