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Assignment 6 10.5.16
http://yojox.com/math412/10-5-16.pdf

We are asked to solve LaPlace's equation for a semi-infinite region:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{matrix} 0 < x < \infty \\ 0 < y < \infty \end{matrix} \tag{1}$$

$$u(0, y) = g(y) \tag{2}$$

$$\frac{\partial}{\partial y} u(x,0) = 0 \tag{3}$$

We will obtain a solution using a Fourier cosine transform in y, since this problem is semi-infinite. Our solution is of the form:

$$u(x, y) = \int_0^\infty \overline{U}(x,\omega)\cos(\omega y)d\omega \tag{4}$$

$$\overline{U} = \frac{2}{\pi}\int_0^\infty u(x, y)\cos(\omega y)dy \tag{5}$$

We substitution $\overline{U}$ into equation (1) to get:

$$\frac{\partial^2 \overline{U}}{\partial x^2} + \frac{\partial^2 \overline{U}}{\partial y^2} = 0 \tag{6}$$

The second derivative of a Fourier Cosine Transform is given in Table 10.5.2 in the Haberman text. Applying it to $\overline{U}$ gives,

$$\frac{\partial^2 \overline{U}}{\partial y^2} = -\frac{2}{\pi}\frac{\partial \overline{U}}{\partial y}(x,0) - \omega^2 \overline{U} \tag{7}$$

But, $\frac{\partial \overline{U}}{\partial y}(x,0) = 0$ from initial data, so

$$\frac{\partial^2 \overline{U}}{\partial y^2} = -\omega^2 \overline{U} \tag{8}$$

Equation (6) now becomes:

$$\frac{\partial^2 \overline{U}}{\partial x^2} - \omega^2 \overline{U} = 0 \quad , \; 0 < x < \infty \tag{9}$$

The boundary conditions for this differential equation are:

$$\overline{U}(0,\omega) = \frac{2}{\pi}\int_0^\infty u(0,y)\cos(\omega y)dy = \frac{2}{\pi}\int_0^\infty g(y)\cos(\omega y)dy \qquad (10)$$

$$\lim_{x\to\infty}\overline{U}(0,\omega) = 0 \qquad (11)$$

We choose the boundary condition in equation (11) because only bounded solutions are of interest to us.

The general solution to this equation would be $\overline{U} = Ae^{\omega x}$, if x and ω were allowed to be negative. However, since they must both be positive, we need to recover the negative exponent terms. The general solution then becomes:

$$\overline{U} = a(\omega)e^{-\omega x} + b(\omega)e^{\omega x} \qquad (12)$$

We may then apply our boundary conditions. First, from equation (11),

$$\lim_{x\to\infty}\overline{U}(0,\omega) = a(\omega)e^{-\infty} + b(\omega)e^{\infty} = 0 + b(\omega)e^{\infty} = 0 \qquad (13)$$

Therefore, $b(\omega) = 0$.

Next, from equation (10):

$$\overline{U}(0,\omega) = a(\omega)e^{0} = \frac{2}{\pi}\int_0^\infty g(y)\cos(\omega y)dy \qquad (14)$$

Therefore,

$$a(\omega) = \frac{2}{\pi}\int_0^\infty g(y)\cos(\omega y)dy \qquad (15)$$

So our solution becomes:

$$\overline{U} = a(\omega)e^{-\omega x}, \qquad (16)$$

where $a(\omega)$ given by equation (15) is the Fourier cosine transform of g(y).

We wish to use the convolution theorem, which states that for a Fourier cosine transform in x, $H(\omega) = F(\omega)G(\omega)$, the product of two Fourier cosine transforms, then

$$h(x) = \frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x-\bar{x}) + f(x+\bar{x})]d\bar{x} \qquad (17)$$

In order to make equation (16) fit into equation (17) we need to find $f(y)$, the Fourier cosine transform of $e^{-\omega x}$.

$$f(y) = \int_0^\infty e^{-\omega x}\cos(\omega y)d\omega \qquad (18)$$

but $\cos(u) = \left(\frac{e^{iu} + e^{-iu}}{2}\right)$. Using this substitution in equation (18) gives:

$$f(y) = \int_0^\infty e^{-\omega x}\left(\frac{e^{i\omega y} + e^{-i\omega y}}{2}\right)d\omega \qquad (19)$$

$$f(y) = \frac{1}{2}\int_0^\infty \left(e^{-\omega(x-iy)} + e^{-\omega(x+iy)}\right)d\omega \qquad (20)$$

$$f(y) = \frac{1}{2}\left[-\frac{e^{-\omega(x-iy)}}{x-iy} - \frac{e^{-\omega(x+iy)}}{x+iy}\right]_0^\infty \qquad (21)$$

$$f(y) = \frac{1}{2}\left[\left(-\frac{e^{-\infty}}{x-iy} - \frac{e^{-\infty)}}{x+iy}\right) - \left(-\frac{e^{0}}{x-iy} - \frac{e^{0}}{x+iy}\right)\right] \qquad (22)$$

$$f(y) = \frac{1}{2}\left(\frac{1}{x-iy} + \frac{1}{x+iy}\right) = \frac{1}{2}\left(\frac{x+iy+x-iy}{(x-iy)(x+iy)}\right) \qquad (23)$$

$$f(y) = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) = \frac{x}{x^2+y^2} \qquad (24)$$

We now use $f(y)$ in equation (17) to get our final solution:

$$u(x,y) = \frac{x}{\pi}\int_0^\infty g(\bar{y})\left[\frac{1}{x^2+(y-\bar{y})^2} + \frac{1}{x^2+(y+\bar{y})^2}\right]d\bar{y} \qquad (25) \blacksquare$$