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Assignment 1 12.4.1
http://yojox.com/math412/12-4-1.pdf

In this problem, we are given a wave equation with homogenous initial conditions and an inhomogeneous boundary condition:

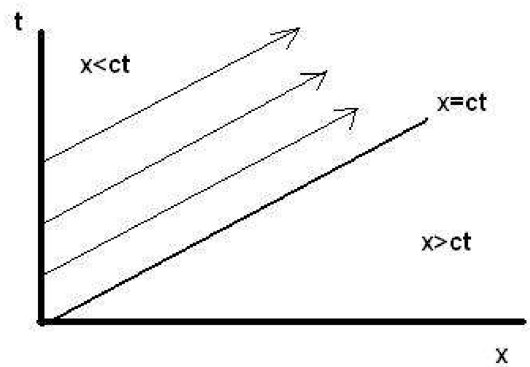
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$u(x,0) = f(x) = 0 \quad (2)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = 0 \quad (3)$$

$$u(0,t) = h(t) \quad (4)$$

This problem must be looked at in two separate regions, $x < ct$ and $x > ct$, as demonstrated in the figure below. We will examine the function in the physical region, such that $x > 0$ and $t > 0$.



For $x > ct$, the function is too far away to have received any information of a boundary condition at x=0 (equation (4)), since this information only travels at the speed of c. As such, we will solve for this region using only equations (1,2,3).

Using d'Alembert's solution to equation (1),

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x}\,, \quad (5)$$

we see that with initial conditions (2) and (3), this simplifies to

$$u(x,t) = 0\,. \quad (6)$$

For $x < ct$, we must make use of our boundary condition, since the function is near enough to receive information of the boundary. To begin, we observe that the dividing line between this case and the previously solved case is $x = ct$. Rearranging this gives $ct - x = 0$, or $t - \frac{x}{c} = 0$.

Next, we take a look at our boundary condition:

$$u(0,t) = h(t) = F(-ct) + G(ct)\,, \tag{7}$$

where the functions F and G are as defined on p.546 of the Haberman text. We need to know how this function behaves when $x < 0$ to know how the waves begin. Here we observe that for $x < ct$,

$$F(-c(-\frac{x}{c})) = F(x) = h(-\frac{x}{c}) \tag{8}$$

$$G(c(\frac{x}{c})) = G(x) = h(\frac{x}{c})\,, \tag{9}$$

for time $t = 0$.

Taking into account t, we see that

$$u(x,t) = h(t - \frac{x}{c})\,. \tag{9}$$

Finally, combining equations (6) and (9) into a piecewise function gives our solution:

$$u(x,t) = \left\{ \begin{matrix} 0, & x > ct \\ h(t - \frac{x}{c}), & x < ct \end{matrix} \right\} \tag{10}$$