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Assignment 4 2.5.1(a,d)
http://yojox.com/math412/2-5-1.pdf

(a) We are asked to solve Laplace's equation $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right)$ inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions:

$$\frac{\partial u}{\partial x}(0, y) = 0 \qquad u(x,0) = 0$$
$$\frac{\partial u}{\partial x}(L, y) = 0 \qquad u(x,H) = f(x)$$

We begin by looking for a solution of the form:

$$u(x, y) = X(x)Y(y)$$

Substituting this into the Laplace equation gives:

$$X''Y + XY'' = 0$$

Then, separating variables yields:

$$\frac{-X''}{X} = \frac{Y''}{Y} = -\lambda^2$$

This gives us two ordinary differential equations:

$$X'' - \lambda^2 X = 0$$
$$Y'' + \lambda^2 Y = 0$$

The solutions to these equations are:

$$X = a_1 \cosh(\lambda x) + a_2 \sinh(\lambda x)$$
$$Y = b_1 \cos(\lambda y) + b_2 \sin(\lambda y)$$

We next apply our boundary conditions:

$$X' = -a_1 \lambda \sinh(\lambda x) + a_2 \lambda \cosh(\lambda x)$$
$$X'(0) = -a_1 \lambda \sinh(0) + a_2 \lambda \cosh(0) = 0$$

So $a_2 = 0$.

$$X'(L) = -a_1 \lambda \sinh(\lambda L) = 0$$

So $\lambda = \frac{n\pi}{L}$.

$$Y(0) = b_1 \cos(0) + b_2 \sin(0) = 0$$

So $b_1 = 0$.

Next we can form our solution as the infinite sum of the product solutions we have obtained:

$$u(x,y) = \sum_{n=0}^{\infty} A_n \cosh\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right).$$

The equation for the coefficient A_n is found with our last boundary value:

$$u(x,H) = \sum_{n=0}^{\infty} A_n \cosh\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi H}{L}\right) = f(x)$$

We find A_n symbolically using the technique of multiplying both sides by $\cosh\left(\frac{m\pi x}{L}\right)$ and integrating over the x domain. All terms are zero except when $m = n$. In this case:

$$\int_0^L f(x)\cosh\left(\frac{n\pi x}{L}\right)dx = A_n\left(\frac{L}{2}\right)\sin\left(\frac{n\pi H}{L}\right).$$

Rearranging, we get

$$A_n = \left(\frac{2}{L}\right)\frac{1}{\sin\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cosh\left(\frac{n\pi x}{L}\right)dx. \quad \blacksquare$$

(d) As in part (a), we are looking at the Laplace equation on a rectangle. This time the boundary conditions are:

$$u(0,y) = g(y) \qquad \frac{\partial u}{\partial y}(x,0) = 0$$

$$u(L,y) = 0 \qquad u(x,H) = 0$$

The separation of variables procedure is the same as part (a), and the resulting differential equations are the same. This time we make a different choice for the solution of X, however.

$$X = a_1\cosh(\lambda(x-L)) + a_2\sinh(\lambda(x-L))$$

$$Y = b_1\cos(\lambda y) + b_2\sin(\lambda y)$$

The solution for X is still valid, and will make it easier to apply our boundary conditions.

$$X(L) = a_1\cosh(0) + a_2\sinh(0) = 0$$

So, a_1=0.

$$Y' = -b_1\lambda\sin(\lambda y) + b_2\lambda\cos(\lambda y)$$
$$Y'(0) = -b_1\lambda\sin(0) + b_2\lambda\cos(0)$$

So, b_2=0.

$$Y(H) = b_1\cos(\lambda H) = 0$$

This equation is satisfied for eigenvalues $\lambda = \frac{(2n+1)\pi}{2H}$.

We can now form our solution.

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sinh\left(\left(\frac{(2n+1)\pi}{2H}\right)(x-L)\right)\cos\left(\frac{(2n+1)\pi}{2H}y\right) \quad \blacksquare$$

We apply our last boundary condition to solve for A_n:

$$u(0,y) = \sum_{n=0}^{\infty} A_n \sinh\left(\left(\frac{(2n+1)\pi}{2H}\right)(-L)\right)\cos\left(\frac{(2n+1)\pi}{2H}y\right) = g(y)$$

As in part (a), we will multiply both sides by $\cos\left(\frac{(2m+1)\pi}{2H}y\right)$ and integrate over the y domain. All terms are zero except when $m = n$. In this case:

$$\int_0^H g(y)\cos\left(\frac{(2n+1)\pi}{2H}y\right)dx = A_n\left(\frac{H}{2}\right)\sinh\left(\left(\frac{(2n+1)\pi}{H}\right)(-L)\right)$$

Rearranging, this gives:

$$A_n = \frac{2\int_0^H g(y)\cos\left(\frac{(2n+1)\pi}{2H}y\right)dy}{H\sinh\left(\left(\frac{(2n+1)\pi}{H}\right)(-L)\right)} \quad \blacksquare$$