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Assignment 9 5.5.1(c)
http://yojox.com/math412/5-5-1.pdf

We are asked to show that the following boundary conditions yield a self-adjoint problem, that is, any two functions u and v satisfying the boundary conditions also satisfy

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b.$$

The boundary conditions for part (c) are:

$$\frac{d\phi}{dx}(0) - h\phi(0) = 0$$
$$\frac{d\phi}{dx}(L) = 0$$

To solve this, we must first write two arbitrary functions that satisfy the boundary conditions:

$$\frac{du}{dx}(0) - hu(0) = 0 \qquad\qquad \frac{dv}{dx}(0) - hv(0) = 0$$
$$\frac{du}{dx}(L) = 0 \qquad\qquad \frac{dv}{dx}(L) = 0$$

Next, we evaluate:

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_0^L$$

$$p(L)\left(u(L)\frac{dv}{dx}(L) - v(L)\frac{du}{dx}(L)\right) - p(0)\left(u(0)\frac{dv}{dx}(0) - v(0)\frac{du}{dx}(0)\right)$$

But from boundary conditions, $\frac{du}{dx}(L) = \frac{dv}{dx}(L) = 0$, so the expression above simplifies to:

$$-p(0)\left(u(0)\frac{dv}{dx}(0) - v(0)\frac{du}{dx}(0)\right)$$

Next we must find $\frac{du}{dx}(0)$ and $\frac{dv}{dx}(0)$ in terms of u and v. We do this by using our boundary conditions:

$$\frac{du}{dx}(0) - hu(0) = 0 \quad \therefore \quad \frac{du}{dx}(0) = hu(0)$$
$$\frac{dv}{dx}(0) - hv(0) = 0 \quad \therefore \quad \frac{dv}{dx}(0) = hv(0)$$

Substituting these back into the equation gives:

$$-p(0)\big(u(0)hv(0)-v(0)hu(0)\big)=0 \quad \blacksquare$$

Therefore, the boundary conditions yield a self-adjoint problem. ■