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Assignment 9 5.5.2
http://yojox.com/math412/5-5-2.pdf

We are asked to prove that the eigenfunctions corresponding to different eigenvalues of the following eigenvalue problem are orthogonal:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right]+q(x)\phi+\lambda\sigma(x)\phi=0$$

with the boundary conditions

$$\phi(1)=0$$
$$\phi(2)-2\frac{d\phi}{dx}(2)=0\,.$$

Also, we must answer: what is the weighting function?

We begin by rewriting the eigenvalue problem using the linear differential operator as defined on p.174 of the Haberman text.

$$L(\phi)+\lambda\sigma(x)\phi=0$$

Next we define λ_n and λ_m as eigenvalues with corresponding eigenfunctions ϕ_n and ϕm . These eigenvalues satisfy:

$$L(\phi_n)+\lambda_n\sigma(x)\phi_n=0$$
$$L(\phi_m)+\lambda_m\sigma(x)\phi_m=0$$

We next apply Green's formula, which is equation (5.5.8) in the Haberman text:

$$\int_1^2[\phi_m L(\phi_n)-\phi_n L(\phi_m)]dx=p(x)\left(\phi_m\frac{d\phi_n}{dx}-\phi_n\frac{d\phi_m}{dx}\right)\Big|_1^2$$

We can simplify the left integral by rearranging our differential equations to isolate *L*.

$$L(\phi_n)=-\lambda_n\sigma(x)\phi_n$$
$$L(\phi_m)=-\lambda_m\sigma(x)\phi_m$$

Substituting these into the above equation gives:

$$\int_1^2[\phi_m(-\lambda_n\sigma(x)\phi_n)-\phi_n(-\lambda_m\sigma(x)\phi_m)]dx=p(x)\left(\phi_m\frac{d\phi_n}{dx}-\phi_n\frac{d\phi_m}{dx}\right)\Big|_1^2$$

$$(\lambda_m-\lambda_n)\int_1^2[\sigma(x)\phi_n\phi_m]dx=p(x)\left(\phi_m\frac{d\phi_n}{dx}-\phi_n\frac{d\phi_m}{dx}\right)\Big|_1^2$$

Next we work on the right hand side, first evaluating the interval:

$$(\lambda_m-\lambda_n)\int_1^2[\sigma(x)\phi_n\phi_m]dx=p(2)\left(\phi_m(2)\frac{d\phi_n}{dx}(2)-\phi_n(2)\frac{d\phi_m}{dx}(2)\right)-p(1)\left(\phi_m(1)\frac{d\phi_n}{dx}(1)-\phi_n(1)\frac{d\phi_m}{dx}(1)\right)$$

But, since $\phi(1)=0$, we can simplify this to:

$$(\lambda_m - \lambda_n)\int_1^2 [\sigma(x)\phi_n\phi_m]dx = p(2)\left(\phi_m(2)\frac{d\phi_n}{dx}(2) - \phi_n(2)\frac{d\phi_m}{dx}(2)\right)$$

Next we make use of our next boundary condition, $\phi(2) - 2\frac{d\phi}{dx}(2) = 0$, by rearranging to isolate $\frac{d\phi}{dx}(2)$:

$$\frac{d\phi}{dx}(2) = \frac{1}{2}\phi(2)$$

And then we substitute this in for both $\frac{d\phi_n}{dx}(2)$ and $\frac{d\phi_m}{dx}(2)$:

$$(\lambda_m - \lambda_n)\int_1^2 [\sigma(x)\phi_n\phi_m]dx = p(2)\left(\phi_m(2)\left(\frac{1}{2}\phi_n(2)\right) - \phi_n(2)\left(\frac{1}{2}\phi_m(2)\right)\right)$$

$$(\lambda_m - \lambda_n)\int_1^2 [\sigma(x)\phi_n\phi_m]dx = 0$$

Since our eigenfunctions are distinct, $\lambda_n \neq \lambda_m$, so the equality is only true when:

$$\int_1^2 \phi_n\phi_m\sigma dx = 0 \quad .$$

Which requires that ϕ_n and ϕ_m be orthogonal with weighting function σ. ■