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Assignment 12 5.9.3
http://yojox.com/math412/5-9-3.pdf

Consider for $\lambda >> 1$

$$\frac{d^2\phi}{dx^2}+[\lambda\sigma(x)+q(x)]\phi=0$$

(a) Substitute

$$\phi = A(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right].$$

Determine a differential equation for $A(x)$.

$$\frac{d^2}{dx^2}\left[A(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]\right]+[\lambda\sigma(x)+q(x)]A(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]=0$$

$$\frac{d}{dx}\left[A'(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]+A(x)i\lambda^{1/2}\sigma^{1/2}(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]\right]$$
$$+[\lambda\sigma(x)+q(x)]A(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]=0$$

$$A''(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]+i\lambda^{1/2}\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]\left(2A'(x)\sigma^{1/2}(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)A(x)\right)$$
$$+qA(x)\exp\left[i\lambda^{1/2}\int_0^x \sigma(x_0)dx_0\right]=0$$

Here we can eliminate all of the exponentials to get to our final equation for $A(x)$:

$$A''(x)+i\lambda^{1/2}\left(2A'(x)\sigma^{1/2}(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)A(x)\right)+qA(x)=0 \quad ▪$$

(b) Let $A(x)=A_0(x)+\lambda^{-1/2}A_1(x)+\ldots$. Solve for $A_0(x)$ and $A_1(x)$. Verify (5.9.8).

We will make the substitution

$$A(x)=\sum_{j=0}^{\infty}\lambda^{-j/2}A_j(x)$$

into the differential equation we obtained in part (a).

$$\sum_{j=0}^{\infty}\lambda^{-j/2}A_j''(x)+i\lambda^{1/2}\left(2\sum_{j=0}^{\infty}\lambda^{-j/2}A_j'(x)\sigma^{1/2}(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)\sum_{j=0}^{\infty}\lambda^{-j/2}A_j(x)\right)+q\sum_{j=0}^{\infty}\lambda^{-j/2}A_j(x)=0$$

$$\sum_{j=0}^{\infty}\lambda^{-j/2}A_j''(x)+i\left(2\sum_{j=0}^{\infty}\lambda^{-(j-1)/2}A_j'(x)\sigma^{1/2}(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)\sum_{j=0}^{\infty}\lambda^{-(j-1)/2}A_j(x)\right)+q\sum_{j=0}^{\infty}\lambda^{-j/2}A_j(x)=0$$

Next, we change the indices so that all terms have $\lambda^{-(j-1)/2}$:

$$\sum_{j=1}^{\infty}\lambda^{-(j-1)/2}A_{j-1}''(x)+i\left(2\sum_{j=0}^{\infty}\lambda^{-(j-1)/2}A_j'(x)\sigma^{1/2}(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)\sum_{j=0}^{\infty}\lambda^{-(j-1)/2}A_j(x)\right)+q\sum_{j=1}^{\infty}\lambda^{-(j-1)/2}A_{j-1}(x)=0$$

To find $A_0(x)$, we take *j=0*.

$$i\left(2\sigma^{1/2}(x)\lambda^{1/2}A_0'(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)\lambda^{-1/2}A_0(x)\right)=0$$

The solution of which is:

$$A_0(x)=c_0\sigma^{-1/4}(x) \quad ▪$$

To find $A_1(x)$, we take *j=1*.

$$A_0''(x)+i\left(2\sigma^{1/2}(x)A_1'(x)+\tfrac{1}{2}\sigma^{-1/2}(x)\sigma'(x)A_1(x)\right)+q(x)A_0(x)=0$$

Using Maple software, and taking $c_0 = 1$, the solution to this was found to be:

$$A_1(x)=\left(\int -\frac{1}{32}\frac{e^{\sqrt{\sigma(x)}}\left(-5i\left(\sigma'(x)\right)^2+4i\sigma''(x)\sigma(x)+8\sigma^{7/4}(x)-16iq(x)\sigma^2(x)\right)}{\sigma^{11/4}(x)}dx\right)e^{-\sqrt{\sigma(x)}} \quad ▪$$

By noting that $A(x)\approx A_0(x)$, we see that

$$\phi(x)\approx\sigma^{-1/4}\exp\left[\pm i\lambda^{1/2}\int_0^x\sigma^{1/2}dx_0\right] \quad ▪$$

This verifies (5.9.8) for *p=1*, which is the case in our problem.

(c) Suppose that $\phi(0)=0$. Use $A_1(x)$ to improve (5.9.9).

If $\phi(0)=0$, we take only the sine functions:

$$\phi(x)=A(x)\sin\left(\lambda^{1/2}\int_0^x\sigma^{1/2}dx_0\right)$$

So to improve (5.9.9) we will use both the $A_0(x)$ and $A_1(x)$ terms:

$$\phi(x)=\sigma^{-1/4}\sin\left(\lambda^{1/2}\int_0^x\sigma^{1/2}dx_0\right)+\lambda^{-1/2}A_1(x)\sin\left(\lambda^{1/2}\int_0^x\sigma^{1/2}dx_0\right) \quad ▪$$

where $A_1(x)$ is defined above.

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(d) Use part (c) to improve (5.9.10) if $\phi(L) = 0$.

$$\lambda \sim \left[n\pi \Big/ \int_0^L \sigma^{1/2} dx_0 \right]^2 \quad ▪$$

(e) Obtain a recursion formula for $A_n(x)$.

$$A_{n+1} = \tfrac{i}{2} \sigma^{-1/4} \int_0^x \sigma^{-1/4} \left(A_n'' + q A_n \right) dx_0 \quad ▪$$