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Assignment 11 7.5.1
http://yojox.com/math412/7-5-1.pdf

The vertical displacement of a nonuniform membrane satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where *c* depends on *x* and *y*. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

(a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

Substituting the separated solution into our PDE gives:

$$\phi h'' = hc^2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

Dividing through by $c^2 h\phi$ gives:

$$\frac{1}{c^2}\frac{h''}{h} = \frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

We set these equal to an arbitrary constant:

$$\frac{1}{c^2}\frac{h''}{h} = \frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = -\lambda$$

We can see that we have separated out our time variable,

$$\frac{1}{c^2}\frac{h''}{h} = -\lambda$$

which gives $h = \sin(c\sqrt{\lambda}t) + \cos(c\sqrt{\lambda}t)$. ▪

Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \text{ with } \phi = 0 \text{ boundary.}$$

From the separation of variables above,

$$\frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = -\lambda\text{, which can be rewritten as } \frac{1}{\phi}\left(\nabla^2\phi\right) = -\lambda$$

Multiplying by ϕ and rearranging gives:

$$\nabla^2\phi + \lambda\phi = 0$$

Which means that solves the ϕ eigenvalue problem, $\nabla^2\phi + \lambda\sigma(x, y)\phi = 0$ with $\phi = 0$ boundary, with $\sigma(x, y) = 1$. ▪

(b) If the eigenvalues are known (and $\lambda \geq 0$), determine the frequencies of vibration.

From our first separation of variables, we found that

$$\frac{1}{c^2}\frac{h''}{h} = -\lambda\text{, with } h \text{ a function of } t.$$

Multiplying both sides of this by hc^2 gives:

$$h'' = -c^2 \lambda h$$

The solution to this is a combination of sines and cosines:

$$h(t) = A\cos\left(c\sqrt{\lambda}\, t\right) + B\sin\left(c\sqrt{\lambda}\, t\right)$$

Which has a frequency of vibration of $c\sqrt{\lambda}$. ▪