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Assignment 11 7.7.8
http://yojox.com/math412/7-7-8.pdf

Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a circle of radius a with the entire boundary insulated, if initially

$$u(r,\theta,0) = f(r,\theta).$$

Briefly analyze $\lim_{t\to\infty} u(r,\theta,t)$. Compare this to what you expect to occur using physical reasoning as $t \to \infty$.

We begin by separating variables, looking for a solution of the form:
$u(r,\theta,t) = \phi(r,\theta)h(t)$.
Substituting the separated solution into our PDE gives:

$$\phi h' = hk\nabla^2\phi$$

$$\frac{1}{k}\frac{h'}{h} = \frac{\nabla^2\phi}{\phi} = -\lambda$$

From which we can get our solution for *h(t)*:

$$h(t) = e^{-\lambda kt}$$

Rearranging the part of the equation that deals with ϕ,

$$\nabla^2\phi + \lambda\phi = 0 .$$

Next, we look for a separated solution of the form

$$\phi(r,\theta) = R(r)\Theta(\theta)$$

Making this substitution gives:

$$\frac{\theta}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} + \lambda R\Theta = 0$$

Multiplying this by $\frac{r^2}{R\Theta}$ gives:

$$r\left(\frac{rR''}{R} + \frac{R'}{R}\right) + \frac{\Theta''}{\Theta} + \lambda r^2 = 0$$

$$\frac{r^2R''}{R} + \frac{rR'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

At this point we can form two equations:

$$r^2R'' + rR' + \left(\lambda r^2 - \mu\right)R = 0 \qquad\qquad \Theta'' = -\mu\Theta$$

We begin by focusing on the equation for Θ. The solution is:

$$\Theta_m = A_m\cos\left(\sqrt{\mu_m}\,\theta\right) + B_m\sin\left(\sqrt{\mu_m}\,\theta\right)$$

The boundary conditions are periodic:

$$\Theta(-\pi) = \Theta(\pi)$$

$$\Theta'(-\pi) = \Theta'(\pi)$$

In order for these conditions to be satisfied, $\sqrt{\mu_m} = m$, and integer. So, the Θ_m solutions are of the form:

$$\Theta_m = A_m \cos(m\theta) + B_m \sin(m\theta)$$

Next we turn to the equation for *R* and its boundary conditions:

$$r^2 R'' + rR' + \left(\lambda r^2 - m^2\right)R = 0$$

$$\frac{dR}{dr}(a) = 0$$

$$|R(0)| < \infty \text{ - boundedness}$$

To remove λ from the equation, we make the change of variable $z = \sqrt{\lambda} r$.

$$\frac{z^2}{\lambda}\frac{d^2R}{dz^2}\left(\frac{dz}{dr}\right)^2 + \frac{z}{\sqrt{\lambda}}\frac{dR}{dz}\frac{dz}{dr} + \left(z^2 - m^2\right)R = 0$$

$$\frac{z^2}{\lambda}\frac{d^2R}{dz^2}\left(\sqrt{\lambda}\right)^2 + \frac{z}{\sqrt{\lambda}}\frac{dR}{dz}\sqrt{\lambda} + \left(z^2 - m^2\right)R = 0$$

$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + \left(z^2 - m^2\right)R = 0$$

The solution to this differential equation is the linear combination of Bessel's functions:

$$R = c_1 J_m(z) + c_2 Y_n(z)$$

For our boundedness condition, we must choose $c_2 = 0$. We also reverse our change of variable to get:

$$R_m = c_1 J_m\left(\sqrt{\lambda} r\right)$$

We apply our other boundary condition to determine λ_{mn}:

$$\frac{dR}{dr}(a) = c_1\sqrt{\lambda_{mn}}\, J'_m\left(\sqrt{\lambda_{mn}}\, a\right) = 0$$

Thus our λ_{mn} are determined by $J'_m\left(\sqrt{\lambda_{mn}}\, a\right) = 0$. ▪

Before assembling our solutions, we must analyze the case when $m = 0$, which makes $\lambda_{mn} = 0$.

$$\Theta_m = A_m \cos(0) + B_m \sin(0) = A_m$$

$$R_m = c_1 J_0\left(\sqrt{\lambda} r\right) \approx 1$$

We reform *R* and Θ into our eigenfunctions:

$$\phi_{mn} = \begin{cases} A_{01} & ;\ \text{m} = 0, \text{n} = 1 \\ A_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right)\cos(m\theta) + B_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right)\sin(m\theta) & ;\ \text{otherwise} \end{cases}$$ ▪

Thus our solution is

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \phi_{mn} e^{-k\lambda_{mn} t}$$ ▪

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Our arbitrary constants can be found from:

$$A_{01} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta) r dr d\theta}{\pi a^2}$$ ▪

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\cos(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r dr d\theta}{\int_{-\pi}^{\pi}\int_0^a \cos(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r dr d\theta}$$ ▪

$$B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\sin(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r dr d\theta}{\int_{-\pi}^{\pi}\int_0^a \sin(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r dr d\theta}$$ ▪