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Assignment 8 Exercise 3
http://yojox.com/math412/Ex-3.pdf

Consider the Green function for the Laplace problem in the upper half plane,

$$G(x-z,y)=\frac{1}{\pi}\frac{y}{(x-z)^2+y^2}$$

(a) Verify that G satisfies Laplace's equation as a function of x and y.
(That is, $\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=0$.)

I will show the derivatives, but the work required to get them in this simple form is too much to reproduce here.

The first step is to re-write G as a function of x and y:

$$G(x,y)=\frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Next, the derivatives:

$$\frac{\partial G}{\partial x}=\frac{-2y}{\pi}\left(\frac{x}{\left(x^2+y^2\right)^2}\right)$$

$$\frac{\partial^2 G}{\partial x^2}=\frac{2y}{\pi}\left(\frac{-1}{\left(x^2+y^2\right)^2}+\frac{4x^2}{\left(x^2+y^2\right)^3}\right)\ \blacksquare$$

$$\frac{\partial G}{\partial y}=\frac{1}{\pi}\left(\frac{1}{x^2+y^2}-\left(\frac{2y^2}{\left(x^2+y^2\right)^2}\right)\right)$$

$$\frac{\partial^2 G}{\partial y^2}=\frac{2y}{\pi}\left(\frac{-3}{\left(x^2+y^2\right)^2}+\left(\frac{4y^2}{\left(x^2+y^2\right)^3}\right)\right)\ \blacksquare$$

Now we write:

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\frac{2y}{\pi}\left(\frac{-1}{\left(x^2+y^2\right)^2}+\frac{4x^2}{\left(x^2+y^2\right)^3}\right)+\frac{2y}{\pi}\left(\frac{-3}{\left(x^2+y^2\right)^2}+\left(\frac{4y^2}{\left(x^2+y^2\right)^3}\right)\right)$$

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\frac{2y}{\pi}\left(\frac{-1}{\left(x^2+y^2\right)^2}+\frac{4x^2}{\left(x^2+y^2\right)^3}-\frac{3}{\left(x^2+y^2\right)^2}+\left(\frac{4y^2}{\left(x^2+y^2\right)^3}\right)\right)$$

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\frac{2y}{\pi}\left(\frac{-4}{\left(x^2+y^2\right)^2}+\frac{4\left(x^2+y^2\right)}{\left(x^2+y^2\right)^3}\right)$$

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\frac{2y}{\pi}\left(\frac{-4}{\left(x^2+y^2\right)^2}+\frac{4}{\left(x^2+y^2\right)^2}\right)$$

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \quad \blacksquare$$

(b) Show that $\int_{-\infty}^{\infty} G(x-z, y)dz = 1$ for each fixed x and y.

This integral looks like: $\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{y}{(x-z)^2 + y^2}dz$, which indicates a possible antiderivative in the form of *arctan(u)*.

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{y}{(x-z)^2 + y^2}dz = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-z)^2 + y^2}dz$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2\left(\frac{(x-z)^2}{y^2} + 1\right)}dz \quad (y \neq 0)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{y\left(\left(\frac{x-z}{y}\right)^2 + 1\right)}dz$$

Next we introduce a change of variable:

$$u = \frac{x-z}{y} \qquad z = -\infty \to u = \frac{x+\infty}{y} = \infty$$

$$du = \frac{-1}{y}dz \qquad z = \infty \to u = \frac{x-\infty}{y} = -\infty$$

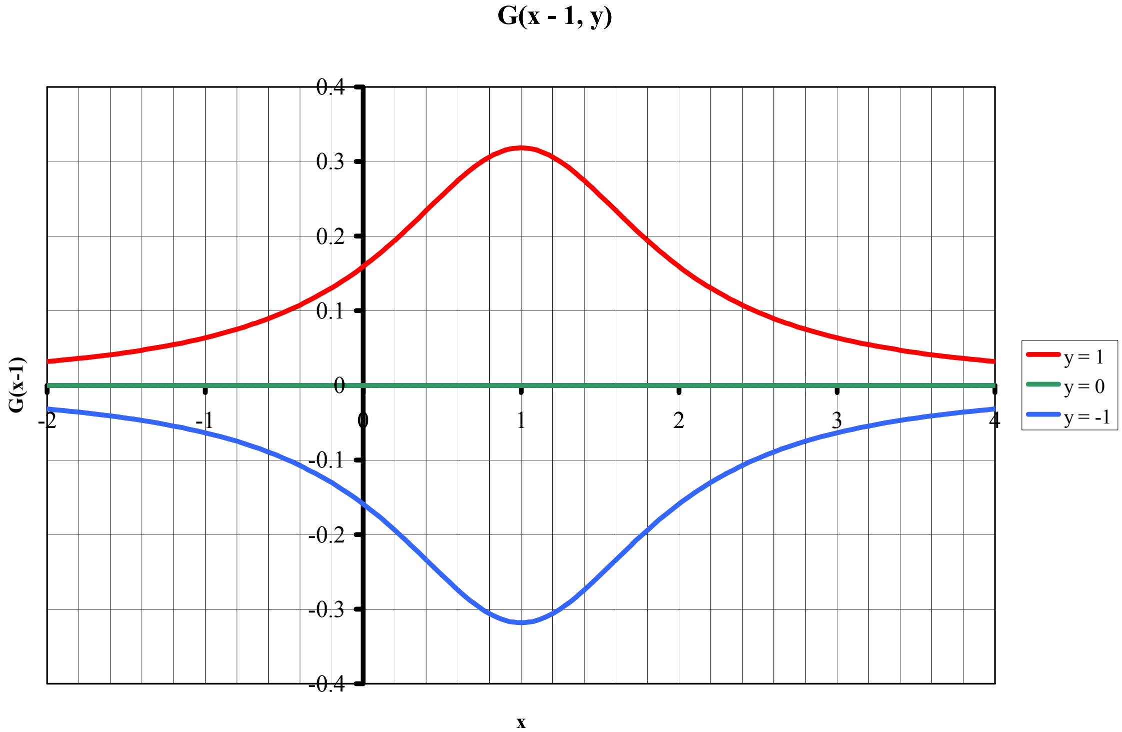
$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{y\left(\left(\frac{x-z}{y}\right)^2 + 1\right)}dz = -\frac{1}{\pi}\int_{\infty}^{-\infty} \frac{1}{u^2+1}du$$

$$= -\frac{1}{\pi}\arctan(u)\Big|_{\infty}^{-\infty}$$

$$= -\frac{1}{\pi}\left(\arctan(-\infty) - \arctan(\infty)\right) = -\frac{1}{\pi}\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = 1 \quad \blacksquare$$

(c) Let $z = 1$ and sketch $G(x-1, y)$ as a function of x for three representative values of y.

The sketch for y values 1, 0, and -1 looks like:



(d) Justify the claim that

$$\lim_{y\to 0^+} G(x-z,y) = \delta(x-z)\,.$$

This means that for any well-behaved function f (say f continuous and bounded),

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} G(x-z,y)f(z)dz = f(x)$$

To begin, I will express the above integral as the sum of three integrals:

$$\lim_{y\to 0^+} \int_{-\infty}^{x-\delta} G(x-z,y)f(z)dz + \lim_{y\to 0^+} \int_{x-\delta}^{x+\delta} G(x-z,y)f(z)dz + \lim_{y\to 0^+} \int_{x+\partial}^{\infty} G(x-z,y)f(z)dz$$

But from our original claim, these three can be re-written as:

$$\int_{-\infty}^{x-\delta} \delta(x-z)f(z)dz + \int_{x-\delta}^{x+\delta} \delta(x-z)f(z)dz + \int_{x+\partial}^{\infty} \delta(x-z)f(z)dz$$

The first and last integrals can be assumed to be zero, since $\delta(x-z)=0$ over these intervals.

The middle interval is next re-written as $f(z) = f(x) + [f(z) - f(x)]$:

$$\int_{x-\delta}^{x+\delta} \delta(x-z) f(x) dz + \int_{x-\delta}^{x+\delta} \delta(x-z)[f(z) - f(x)] dz$$

If *f* is continuous (which we said it is), then the integral on the right approaches zero as δ is chosen smaller and smaller. This just leaves,

$\int_{x-\delta}^{x+\delta} \delta(x-z) f(x) dz = f(x)$, by the properties of the delta function. ■

(e) Argue that *G* is the correct Green function for the problem.

We know from part (a) that *G* satisfies Laplace's equation as a function of *x* and *y,* it follows that multiplying by a function of *z* would still be a solution. The integral over the real line obtains the result of that multiplication at all points. In part (d) we have shown that the initial data $u(x,0) = f(x)$ is satisfied by showing the behavior of our solution as $y \to 0^+$. ■