Jason Hill yojo.ex
Assignment 7 9.3.6(a,c)
http://yojox.com/math412/9-3-6.pdf

We are asked to consider:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0) \quad (1)$$

$$G(0, x_0) = 0 \quad (2) \qquad \frac{dG}{dx}(L, x_0) = 0 \quad (3)$$

(a) To solve for G, we first note that because of the delta function's definition, the second partial derivative in x must be 0 for all $x = x_0$. This means that G must be linear, but the constants are not necessarily the same. This is written as follows:

$$G = \begin{cases} a + bx & x < x_0 \\ c + dx & x > x_0 \end{cases} \quad (4)$$

Next we apply our boundary conditions.

Equation (2) applies to G for $x < x_0$:

$$G(0, x_0) = 0 = a + b * 0 \quad (5)$$

Therefore, $a = 0$.

Equation (3) applies to G for $x > x_0$.

$$G'(x, x_0) = d \quad (6)$$

$$G'(L, x_0) = 0 = d \quad (7)$$

Therefore, $d = 0$.

At this point our Green's function looks like:

$$G = \begin{cases} bx & x < x_0 \\ c & x > x_0 \end{cases} \quad (8)$$

The last two constants can be eliminated by using the two properties of the Green's function.
First, the Green's function is continuous at $x = x_0$:

$$G(x_{0-}, x_0) = G(x_{0+}, x_0) \quad (9)$$

Second, the derivative of the Green's function as a jump discontinuity at $x = x_0$:

$$\left.\frac{dG}{dx}\right|_{x = x_0 +} - \left.\frac{dG}{dx}\right|_{x = x_0 -} = 1 \quad (10)$$

Using equation (9) gives:

$$bx_0 = c \quad (11)$$

Using equation (10) gives

$$0 - b = 1 \quad (12)$$

Solving these two equations gives:

$$b = -1 \quad (13)$$

$$c = -x_0 \quad (14)$$

We substitute this back into equation (8) to get our final solution:

$$G = \begin{cases} -x & x < x_0 \\ -x_0 & x > x_0 \end{cases} \quad (15) \ \blacksquare$$

(c) Next, we are to show that this Green's function satisfies the solution to 9.3.5. Specifically, we need to show:

$$u = \int_0^L f(x_0) G(x, x_0) dx_0 = \int_0^x (x - x_0) f(x_0) dx_0 - x \int_0^L f(x_0) dx_0 \quad (16)$$

We begin by substituting equation (15) into equation (16):

$$\int_0^L f(x_0) G(x, x_0) dx_0 = \int_0^L dx_0 f(x_0) \begin{cases} -x & x < x_0 \\ -x_0 & x > x_0 \end{cases} \quad (17)$$

The integral on the right should be split into two integrals:

$$\int_0^L dx_0 f(x_0) \begin{cases} -x & x < x_0 \\ -x_0 & x > x_0 \end{cases} = \int_0^x -x_0 f(x_0) dx_0 + \int_x^L -x f(x_0) dx_0 \quad (18)$$

Next we add and subtract $x \int_0^x f(x_0) dx_0$. This gives:

$$= \int_0^x -x_0 f(x_0) dx_0 - x \int_x^L f(x_0) dx_0 + x \int_0^x f(x_0) dx_0 - x \int_0^x f(x_0) dx_0 \quad (19)$$

Rearranging yields:

$$= x \int_0^x f(x_0) dx_0 + \int_0^x -x_0 f(x_0) dx_0 - x \int_x^L f(x_0) dx_0 - x \int_0^x f(x_0) dx_0 \quad (20)$$

$$= x\int_0^x f(x_0)dx_0 + \int_0^x -x_0 f(x_0)dx_0 - \left(x\int_x^L f(x_0)dx_0 + x\int_0^x f(x_0)dx_0 \right) \quad (21)$$

The first two integrals and the last two integrals can each be combined as follows:

$$u = \int_0^x (x - x_0) f(x_0)dx_0 - x\int_0^L f(x_0)dx_0 \qquad (22) \ \blacksquare$$