Angela Sung arsung
Assignment 6 Exercise 10.3.5
http://people.tamu.edu/~arsung

If $F(\omega)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. This result is known as the shift theorem for Fourier transforms.

If we prove that the Fourier transform of $f(x-\beta)$ is $e^{i\omega\beta}F(\omega)$, that will be sufficient to show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$.

Let us first define the Fourier transform as

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$$f(x)=\int_{-\infty}^{\infty} e^{-i\omega x} F(\omega)\,d\omega$$

Taking the Fourier transform of $f(x-\beta)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} f(x-\beta)\,dx$$

We can now make a change of variables. Letting $y=x-\beta$, then the integral can be rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(y+\beta)} f(y)\,dy$$

Technically, the limits of integration must be changed to account for the changed variable of integration, but since the limits are $-\infty$ to ∞, they are not affected by subtracting β.

The sum in the exponent can be rewritten as a product of two exponentials:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega y}e^{i\omega\beta} f(y)\,dy$$

Since the term $e^{-i\omega\beta}$ is a constant with respect to *x*, we can pull it out of the integral.

$$e^{i\omega\beta}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega y} f(y)\,dy}_{F(\omega)} = e^{i\omega\beta}F(\omega)$$

We recognize the integral as simply the definition of the Fourier transform, so the Fourier transform of $f(x-\beta)$ is $e^{i\omega\beta}F(\omega)$.