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Assignment 6 Exercise 10.4.8
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Solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \begin{matrix} 0<x<L \\ -\infty<y<\infty \end{matrix}$$

subject to

$$u(0,y)=g_1(y) \qquad u(L,y)=g_2(y)$$

We first take the Fourier transform in y of the entire problem according to the equations

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\hat{f}(\omega)\,d\omega$$

The PDE becomes

$$\frac{\partial^2 \hat{u}(x,\omega)}{\partial x^2}+(i\omega)^2\,\hat{u}(x,\omega)=0$$

$$\frac{\partial^2 \hat{u}}{\partial x^2}=\omega^2\hat{u}(x,\omega)$$

The boundary conditions similarly transform to

$$\hat{u}(0,\omega)=\hat{g}_1(\omega)=\int_{-\infty}^{\infty} e^{-i\omega y} g_1(y)\,dy$$

$$\hat{u}(L,\omega)=\hat{g}_2(\omega)=\int_{-\infty}^{\infty} e^{-i\omega y} g_2(y)\,dy$$

The PDE has the general solution

$$\hat{u}(x,\omega)=C_1(\omega)e^{\omega x}+C_2(\omega)e^{-\omega x} \qquad (1)$$

This solution can also be expressed in terms of $\cosh(\omega x)$ and $\sinh(\omega x)$ since

$$\sinh x=\frac{e^x-e^{-x}}{2} \qquad \cosh x=\frac{e^x+e^{-x}}{2}$$

So we can rewrite Equation (1) as

$$\hat{u}(x,\omega)=a(\omega)\cosh(\omega x)+b(\omega)\sinh(\omega x) \qquad (2)$$

Applying the boundary conditions would be more convenient if the equation contained the function $\sinh(\omega(L-x))$ instead of $\cosh(\omega x)$.

Hyperbolic sine obeys the hyperbolic identity

$$\sinh(\omega L-\omega x)=\sinh(\omega L)\cosh(\omega x)-\cosh(\omega L)\sinh(\omega x)$$

so it is easy to see that a function of $\cosh(\omega x)$ and $\sinh(\omega x)$ can easily be rewritten as a function of $\sinh(\omega(L-x))$ and $\sinh(\omega x)$. We can write Equation (2) as

$$\hat{u}(x,\omega)=A(\omega)\sinh(\omega x)+B(\omega)\sinh(\omega(L-x)) \tag{3}$$

Now applying the boundary conditions to Equation (3), we get

$$\hat{u}(0,\omega)=\cancel{A(\omega)\sinh(0)}+B(\omega)\sinh(\omega L)=\hat{g}_1(\omega)$$

$$\hat{u}(x,\omega)=A(\omega)\sinh(\omega L)+\cancel{B(\omega)\sinh(0)}=\hat{g}_2(\omega)$$

Therefore,

$$B(\omega)=\frac{\hat{g}_1(\omega)}{\sinh(\omega L)} \qquad A(\omega)=\frac{\hat{g}_2(\omega)}{\sinh(\omega L)}$$

Substituting these values into Equation (3),

$$\hat{u}(x,\omega)=\frac{\hat{g}_2(\omega)}{\sinh(\omega L)}\sinh(\omega x)+\frac{\hat{g}_1(\omega)}{\sinh(\omega L)}\sinh(\omega(L-x)) \tag{4}$$

To obtain the original function, we must take the inverse Fourier transform of Equation (4)

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega x}\hat{u}(x,\omega)\,d\omega$$

$$\hat{u}(x,\omega)=\frac{\hat{g}_2(\omega)}{\sinh(\omega L)}\sinh(\omega x)+\frac{\hat{g}_1(\omega)}{\sinh(\omega L)}\sinh(\omega(L-x))$$

$$\hat{g}_1(\omega)=\int_{-\infty}^{\infty}e^{-i\omega y}g_1(y)\,dy$$

$$\hat{g}_2(\omega)=\int_{-\infty}^{\infty}e^{-i\omega y}g_2(y)\,dy$$