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Assignment 1 Exercise 12.3.6
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**Consider the three-dimensional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

**Assume that the solution is spherically symmetric so that**

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right)$$

**a) Make the transformation** $u = \frac{1}{\rho} w(\rho, t)$ **and verify that**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial \rho^2}$$

Originally, we have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right)$$

Substituting $u(x,t) = \frac{1}{\rho} w(\rho, t)$, we get

$$\frac{\partial^2}{\partial t^2}\left(\frac{w(\rho,t)}{\rho}\right) = c^2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial}{\partial \rho}\left(\frac{w(\rho,t)}{\rho}\right)\right]$$

$$\frac{1}{\rho}\frac{\partial^2 w}{\partial t^2} = c^2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\frac{\partial w}{\partial \rho}\rho - w}{\rho^2}\right]$$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\frac{\partial w}{\partial \rho}\rho - w\right]$$

$$= c^2 \frac{1}{\rho}\left(\frac{\partial^2 w}{\partial \rho^2}\rho + \frac{\partial w}{\partial \rho} - \frac{\partial w}{\partial \rho}\right)$$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial \rho^2}$$

**b) Show that the most general spherically symmetric solution of the wave equation consists of the sum of two spherically symmetric waves, one moving outward at speed *c* and the other inward at speed *c*. Note the decay of the amplitude.**

Let $B(\rho - ct)$ be a wave moving inward at speed *c* and $C(\rho + ct)$ be a wave moving outward at speed *c*. For *B* and *C* to be solutions, they must satisfy the wave equation.

$$\frac{\partial^2}{\partial t^2}\left[B(\rho - ct)\right] = \frac{\partial}{\partial t}\left[-cB'(\rho - ct)\right]$$

$$= c^2 B''(\rho - ct)$$

$$c^2 \frac{\partial^2}{\partial \rho^2}\left[B(\rho - ct)\right] = c^2 B''(\rho - ct)$$

$$\therefore \frac{\partial^2}{\partial t^2}\left[B(\rho - ct)\right] = c^2 \frac{\partial^2}{\partial \rho^2}\left[B(\rho - ct)\right]$$

$$\frac{\partial^2}{\partial t^2}\left[C(\rho + ct)\right] = \frac{\partial}{\partial t}\left[cC'(\rho + ct)\right]$$

$$= c^2 C''(\rho + ct)$$

$$c^2 \frac{\partial^2}{\partial \rho^2}\left[C(\rho - ct)\right] = c^2 C''(\rho + ct)$$

$$\therefore \frac{\partial^2}{\partial t^2}\left[C(\rho + ct)\right] = c^2 \frac{\partial^2}{\partial \rho^2}\left[C(\rho + ct)\right]$$

Both $B(\rho - ct)$ and $C(\rho + ct)$ satisfy the wave equation for spherically symmetric solutions. However, to show that the most general solution consists of a sum of these two functions, we must prove that they are the only solutions to the wave equation.

Let $z = \rho - ct$ and $y = \rho + ct$.

$$\frac{\partial z}{\partial \rho} = 1 \qquad \frac{\partial y}{\partial \rho} = 1 \qquad \frac{\partial z}{\partial t} = -c \qquad \frac{\partial y}{\partial t} = c$$

$$\frac{\partial}{\partial \rho} = \frac{\partial z}{\partial \rho}\frac{\partial}{\partial z} + \frac{\partial y}{\partial \rho}\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \qquad \frac{\partial^2}{\partial \rho^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial y} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = -c\frac{\partial}{\partial z} + c\frac{\partial}{\partial y} \qquad \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial z^2} - 2c^2\frac{\partial^2}{\partial z \partial y} + c^2\frac{\partial^2}{\partial y^2}$$

The wave equation $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial \rho^2}$ implies that $\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial \rho^2} = 0$

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial \rho^2} &= c^2 \frac{\partial^2 w}{\partial z^2} - 2c^2 \frac{\partial^2 w}{\partial z \partial y} + c^2 \frac{\partial^2 w}{\partial y^2} - c^2 \left[ \frac{\partial^2 w}{\partial z^2} + 2 \frac{\partial^2 w}{\partial z \partial y} + \frac{\partial^2 w}{\partial y^2} \right] \\ &= -4c^2 \frac{\partial^2 w}{\partial z \partial y} = 0\end{aligned}$$

Therefore, $\frac{\partial^2 w}{\partial z \partial y} = 0$. This equation implies that $\frac{\partial w}{\partial y}$ is a constant with respect to *z* so $\frac{\partial w}{\partial y} = \gamma(y)$, a function of *y* only.

The solution *w* can be found by integrating $\gamma(y)$ with respect to y.

$$\begin{aligned}w(z, y) &= \int_{y_0}^{y} \gamma(\tilde{y})\, d\tilde{y} + B(z) \\ &= C(y) + B(z) \\ &= B(\rho - ct) + C(\rho + ct)\end{aligned}$$

The solution *u* is then

$$u = \frac{1}{\rho} w(\rho, t) = \frac{1}{\rho}\left[ B(\rho - ct) + C(\rho + ct) \right]$$

This solution shows that the general solution to the wave equation is the sum of two spherically symmetric waves, one moving inward and one moving outward. It also shows that the amplitude of the wave decays as $1/\rho$, where $\rho$ is the distance of the wave from the origin. As a wave moves outward, its amplitude becomes smaller. Similarly, as a wave moves inward, its amplitude becomes larger.