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Assignment 1 Exercise 12.4.4
http://physics.tamu.edu/people/showpeople.php?name=Angela%20R.%20Sung&userid=arsung

a) Solve for $x > 0$, $t > 0$ (using the method of characteristics):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\} x > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad t > 0$$

(Assume that u is continuous at $x = 0$, $t = 0$.)

We know that the general solution to the wave equation is $u(x,t) = B(x-ct) + C(x+ct)$.

Therefore, $f(x) = u(x,0) = B(x) + C(x)$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = -cB'(x) + cC'(x)$$

Let $G(x) = \int_0^x g(\tilde{x})\,d\tilde{x} = -cB(x) + cC(x)$ so that $\frac{G(x)}{c} = -B(x) + C(x)$

Adding $f(x)$ and $\frac{G(x)}{c}$, we get $f(x) + \frac{G(x)}{c} = 2C(x)$ so

$$C(x) = \frac{f(x)}{2} + \frac{G(x)}{2c}$$

Similarly, subtracting $\frac{G(x)}{c}$ from $f(x)$ yields $f(x) - \frac{G(x)}{c} = 2B(x)$ and

$$B(x) = \frac{f(x)}{2} - \frac{G(x)}{2c}$$

Therefore, we can conclude that

$$u(x,t) = B(x-ct) + C(x+ct) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x}$$

However, since $f(x)$ and $g(x)$ are only defined for $x > 0$, the equation for $B(x-ct)$ only holds when the argument of $x - ct > 0$ or $x > ct$.

The argument $x + ct$ is always positive since $x > 0$ and $t > 0$, so the equation for $C(x+ct)$ is valid for both $x > ct$ and $x < ct$.

To define $B(x-ct)$ for $x < ct$, we have to use the boundary condition $\frac{\partial u}{\partial x}(0,t)=0$.

$$\frac{\partial u}{\partial x}(0,t)=B'(-ct)+C'(ct)=0$$

$$\int_0^t B'(-c\tilde{t})\,d\tilde{t}=-\int_0^t C'(c\tilde{t})\,d\tilde{t}$$

$$\frac{B(-ct)}{-c}=\frac{-C(ct)}{c} \qquad \Rightarrow B(-ct)=C(ct)$$

$$\Rightarrow B(z)=C(-z),\ z<0$$

Therefore, $B(x-ct)=C(ct-x)$ and

$$\begin{aligned} u(x,t) &= C(ct-x)+C(x+ct) \\ &= \frac{f(ct-x)+f(x+ct)}{2}+\frac{1}{2c}\left[\int_0^{ct-x} g(\tilde{x})\,d\tilde{x}+\int_0^{x+ct} g(\tilde{x})\,d\tilde{x}\right] \qquad \text{for } x<ct. \end{aligned}$$

b) Show that the solution of part (a) may be obtained by extending the initial position and velocity as even functions (around *x* = 0).

If we extend the initial position and velocity functions $f(x)$ and $g(x)$ as even functions around $x = 0$, we have

$$f(-x)=f(x)$$

$$g(-x)=g(x)$$

$$G(-x)=\int_0^{-x} g(\tilde{x})\,d\tilde{x}=-\int_0^{x} g(\tilde{x})\,d\tilde{x}=-G(x)$$

So the solution for $x > ct$,

$$u(x,t)=\frac{f(x-ct)+f(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tilde{x})\,d\tilde{x}$$

correlates with the equation

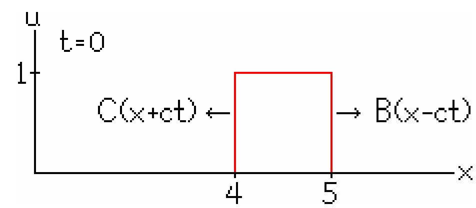
$$\begin{aligned} u(x,t) &= \frac{f(ct-x)+f(x+ct)}{2}+\frac{1}{2c}\left[\int_0^{x+ct} g(\tilde{x})\,d\tilde{x}-\int_{ct-x}^{0} g(\tilde{x})\,d\tilde{x}\right] \\ &= \frac{f(ct-x)+f(x+ct)}{2}+\frac{1}{2c}\left[\int_0^{x+ct} g(\tilde{x})\,d\tilde{x}+\int_0^{ct-x} g(\tilde{x})\,d\tilde{x}\right] \qquad \text{for } x<ct. \end{aligned}$$

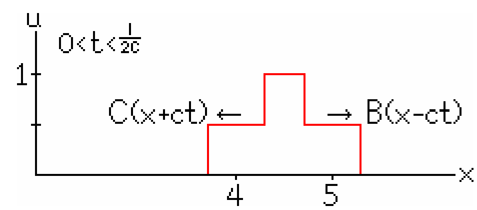
This solution is the same as we got in part (a), so when given Neumann boundary conditions, it is possible to derive the solution for negative arguments, mainly $x - ct < 0$, by extending the initial position and velocity functions f and g as even functions.

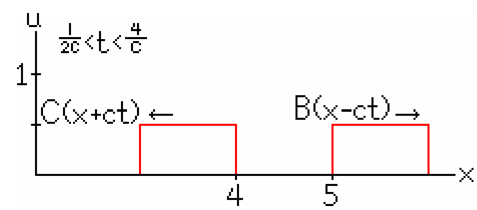
c) Sketch the solution if $g(x) = 0$ and

$$f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

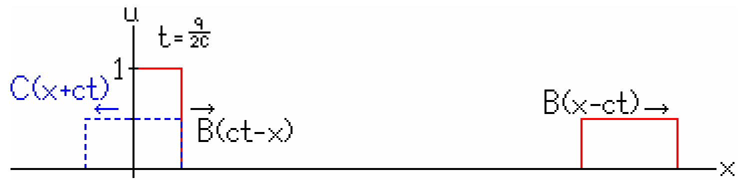
At time $t = 0$, the solution has the form $u(x,0) = \dfrac{2f(x)}{2} = 1$, $4 < x < 5$. As time elapses, the two wave packets move away from each other, one to the right, $B(x-ct)$, and the other to the left, $C(x+ct)$.

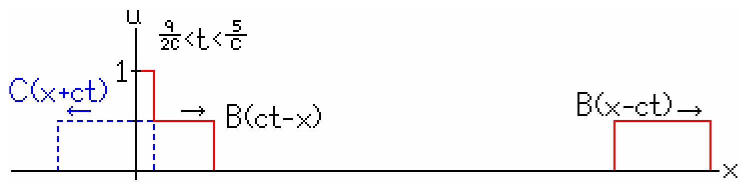




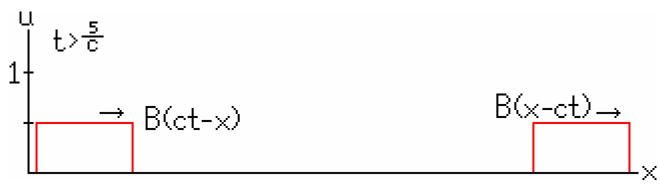


The wave $B(x-ct)$ will continue to travel to the right for all $t > 0$ since there is no boundary to the right. The wave $C(x+ct)$ will hit the boundary $x = 0$ at time $t = \dfrac{4}{c}$. It is reflected back without turning upside down because of the Neumann boundary conditions. The reflected wave travels to the right and satisfies the condition $x < ct$ and so the solution has the form $u(x,t) = \dfrac{f(ct-x)}{2}$. During the time $\dfrac{4}{c} < t < \dfrac{5}{c}$, the original wave $C(x+ct)$ and the reflected wave $B(ct-x)$ interact as shown below:





For time $t > \frac{5}{c}$, the original wave disappears into the imaginary part of the graph, beyond the boundary. The remaining wave is the reflected wave that travels to the right, $B(ct-x)$.



The path of the waves is shown below: