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Assignment 4 Exercise 2.4.2
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Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

We can define $u(x,t) = X(x)T(t)$. This gives us

$$\frac{\partial u}{\partial t} = X(x)T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

So the heat equation is equal to

$$X(x)T'(t) = kX''(x)T(t)$$

Using separation of variables, we get

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} \qquad \text{or} \qquad \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

Since the left side consists of only functions of *t* and the right side consists of only functions of *x*, we know that both sides must equal the same constant. We can call this constant $-\lambda$.

Therefore, separating the equation into two equations, each in one variable, we get

$$\frac{T'(t)}{kT(t)} = -\lambda \qquad \frac{X''(x)}{X(x)} = -\lambda$$

$$T'(t) = -\lambda kT(t) \qquad X''(x) = -\lambda X(x)$$

We know that the general solutions to these equations are

$$T(t) = ce^{-\lambda kt}$$

$$X(x) = a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right)$$

So $u(x,t) = ce^{-\lambda kt}\left(a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right)\right)$

However, the boundary condition $\frac{\partial u}{\partial x}(0,t) = 0$ tells us that $b = 0$ so the sin term disappears and we are left with

$$u(x,t) = Ce^{-\lambda kt}\cos\left(\sqrt{\lambda}x\right)$$

where $C = ca$.

The second boundary condition $u(L,t) = 0$ tells us that

$$u(L,t) = Ce^{-\lambda kt}\cos\left(\sqrt{\lambda}L\right) = 0$$

This is only true if $\sqrt{\lambda} = \frac{\left(n - \frac{1}{2}\right)\pi}{L}$. Therefore,

$$u(x,t) = Ce^{-\left(\frac{\left(n-1/2\right)\pi}{L}\right)^2 kt}\cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{L}x\right)$$

This equation is the solution the PDE and the BC for any *n* that is a positive integer. The general solution to the equation is therefore a sum of all of these solutions.

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{\left(n-1/2\right)\pi}{L}\right)^2 kt}\cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{L}x\right)$$

Applying the initial condition $u(x,0) = f(x)$ gives

$$\sum_{n=1}^{\infty} C_n \cos\left(\frac{\left(n - \frac{1}{2}\right)\pi}{L}x\right) = f(x)$$

We can find the equation for C_n by multiplying both sides of the equation by $\cos(mx)$ and integrating over the interval $[-L, L]$.

$$\sum_{n=1}^{\infty} C_n \int_{-L}^{L} \cos\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)\cos(mx)\,dx = \int_{-L}^{L} f(x)\cos(mx)\,dx$$

The orthogonality relation

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases}$$

tells us that if $m \neq n-\frac{1}{2}$, then the left side of the equation equals zero. We only need to worry about the case where $m = n-\frac{1}{2}$ and since the summation defines *n* from 1 to ∞, $n-\frac{1}{2}$ is never equal to zero. This only leaves the case where $m = n-\frac{1}{2} \neq 0$, for which the integral

$$\int_{-L}^{L} \cos\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)\cos(mx)\,dx = 2\int_0^L \cos\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)\cos(mx)\,dx = 2\frac{L}{2} = L$$

Therefore,

$$\begin{aligned} C_n L &= \int_{-L}^{L} f(x)\cos(mx)\,dx \\ &= 2\int_0^L f(x)\cos\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)dx \end{aligned}$$

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)dx$$

So the solution to the problem is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{(n-1/2)}{L}kt}\cos\left(\frac{(n-\frac{1}{2})}{L}x\right)$$

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(n-\frac{1}{2})\pi}{L}x\right)dx$$