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Assignment 10 Exercise 2.5.6(a)
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Solve Laplace's equation inside a semicircle of radius a ($0 < r < a,\ 0 < \theta < \pi$) subject to the boundary conditions

u = 0 on the diameter

$$u(a,\theta) = g(\theta)$$

Laplace's equation in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Using separation of variables,

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Dividing the equation by $\frac{R\Theta}{r^2}$ gives

$$\frac{(rR')'r}{R} + \frac{\Theta''}{\Theta} = 0$$

$$\frac{(rR')'r}{R} = -\frac{\Theta''}{\Theta} = k^2$$

We get two equations:

$$r^2R'' + rR' - k^2R = 0 \qquad \Theta'' = -k^2\Theta$$

Taking the equation in r first,

$$R'' + \frac{1}{r}R' - \frac{k^2}{r^2}R = 0$$

We try solutions of the form $R = r^p$. Substituting into the equation,

$$p(p-1)r^{p-2} + pr^{p-2} - k^2r^{p-2} = 0$$

$$r^{p-2}\left(p^2 - p + p - k^2\right) = 0$$

$$p^2 - k^2 = 0$$

$$p = \pm k$$

The general solution for R is

$$R(r) = cr^k + dr^{-k}$$

For the special case of $k = 0$, the general solution is

$$R(r) = c_0 + d_0 \ln r$$

However, if we look at these equations, we see that for k positive, $r^k \to 0$ as $r \to 0$, but $r^{-k} \to \pm\infty$ as $r \to 0$. Therefore, considering only bounded solutions, $d = d_0 = 0$, and we have

$$R(r) = cr^k \qquad\qquad k \geq 0$$

The equation in θ gives the solution

$$\Theta(\theta) = A\cos k\theta + B\sin k\theta$$

The first boundary condition, $u = 0$ on the diameter, tells us

$$\Theta(0) = 0 \qquad\qquad\qquad \Theta(\pi) = 0$$

Applying these conditions, $A = 0$, so we are left with

$$\Theta(\theta) = B\sin k\theta$$

Thus the general solution for u is

$$u(r,\theta) = \sum_{k=1}^{\infty} B_k r^k \sin k\theta$$

Applying the second boundary condition, $u(a,\theta) = g(\theta)$, gives

$$\sum_{k=1}^{\infty} B_k a^k \sin k\theta = g(\theta)$$

Multiplying both sides by $\sin k\theta$ and integrating over θ for $0 < \theta < \pi$,

$$B_k = \frac{2}{\pi a^k} \int_0^{\pi} g(\theta) \sin k\theta d\theta$$

So the final solution is

$$u(r,\theta) = \sum_{k=1}^{\infty} B_k r^k \sin k\theta$$

$$B_k = \frac{2}{\pi a^k} \int_0^{\pi} g(\theta) \sin k\theta d\theta$$