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Assignment 3 Exercise 3.3.16
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Fourier series can be defined on other intervals besides $-L \le x \le L$. Suppose that $g(y)$ is defined for $a \le y \le b$. Represent $g(y)$ using periodic trigonometric functions with period $b-a$. Determine formulas for the coefficients.
[Hint: Use the linear transformation $y = \frac{a+b}{2} + \frac{b-a}{2L}x$.]

Starting with the Fourier series of $f(x)$ defined on $[-L, L]$, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

We can now make a substitution of $g(y)$ for $f(x)$. In order to do this, we also need to change the argument of the sine and cosine functions, as well as the limits of integration and the constants $\frac{1}{2L}$ and $\frac{1}{L}$ in the equations for the coefficients.

To change the arguments of the sine and cosine functions, we use the linear transformation

$$y = \frac{a+b}{2} + \frac{b-a}{2L}x$$

or

$$x = \frac{2L}{b-a}\left(y - \frac{a+b}{2}\right) = L\left(\frac{2}{b-a}y - \frac{a+b}{b-a}\right)$$

It should be noted that the given transformation does indeed shift the interval from $[-L, L]$ to $[a,b]$ since

$$\text{for } x=-L,\ y=\frac{a+b}{2}+\frac{b-a}{2\cancel{L}}\left(-\cancel{L}\right)=\frac{a+b-(b-a)}{2}=\frac{2a}{2}=a$$

$$\text{for } x=L,\ y=\frac{a+b}{2}+\frac{b-a}{2\cancel{L}}\left(\cancel{L}\right)=\frac{a+b+(b-a)}{2}=\frac{2b}{2}=b$$

The limits of integration are easily transformed since then are the just the endpoints of the interval. Instead of integrating from $-L$ to L, we integrate from a to b.

The constants in the coefficient equations are also easily changed. For the interval $[-L, L]$, the period is $2L$ so $\frac{1}{2L}$ represents the inverse of the period. Similarly, $\frac{1}{L}$ is the inverse of half the period. In the case of $[a,b]$, the period is $b-a$, so $\frac{1}{2L}$ becomes $\frac{1}{b-a}$ and $\frac{1}{L}$ becomes $\frac{2}{b-a}$.

Thus we can rewrite the Fourier series for a function defined on $[a,b]$ as

$$g(y)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2n\pi}{b-a}y-\frac{n\pi(a+b)}{b-a}\right)+b_n\sin\left(\frac{2n\pi}{b-a}y-\frac{n\pi(a+b)}{b-a}\right)\right]$$

$$a_0=\frac{1}{b-a}\int_a^b g(y)\,dy$$

$$a_n=\frac{2}{b-a}\int_a^b g(y)\cos\left(\frac{2n\pi}{b-a}y-\frac{n\pi(a+b)}{b-a}\right)dy$$

$$b_n=\frac{2}{b-a}\int_a^b g(y)\sin\left(\frac{2n\pi}{b-a}y-\frac{n\pi(a+b)}{b-a}\right)dy$$