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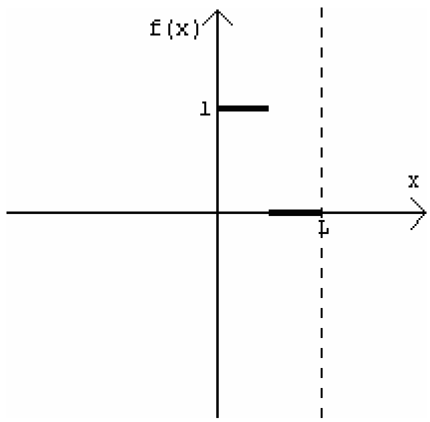
Assignment 3 Exercise 3.3.3(b,c)
http://physics.tamu.edu/people/showpeople.php?name=Angela%20R.%20Sung&userid=arsung

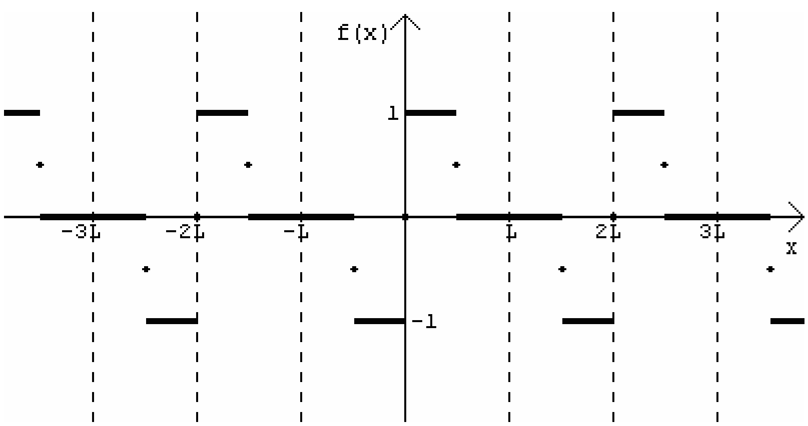
For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:

(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

The graph of the function for $0 \le x \le L$ is shown below.



The Fourier sine series is the odd periodic extension of the function. At discontinuities, the Fourier sine series takes on values of the function that are halfway between the two values of the function



Solving for the Fourier sine series solution, we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^{L/2}\sin\frac{n\pi x}{L}\,dx + \cancel{\frac{2}{L}\int_{L/2}^{L} 0\,dx}$$

$$= -\frac{2}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/2} = -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right)-1\right)$$

$$= \begin{cases} \dfrac{2}{n\pi} & \text{for } n \text{ odd} \\ \dfrac{2}{n\pi}\left(\left(-1\right)^{n/2+1}+1\right) & \text{for } n \text{ even} \end{cases}$$

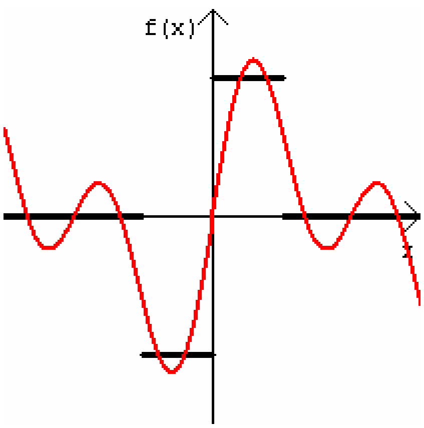
So the series can be written as

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{L} + \frac{1}{n\pi}\left(\left(-1\right)^{n+1}+1\right)\sin\frac{2n\pi x}{L}$$

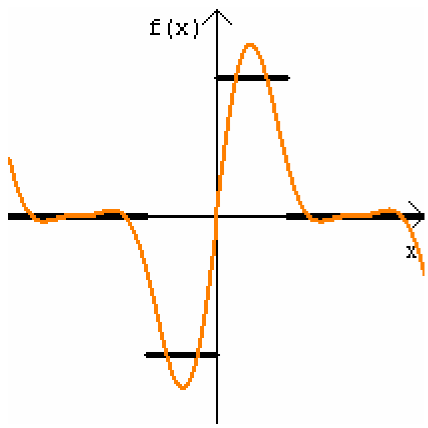
Below are some graphs of the finite sums of the Fourier sine series

$$f(x) = \sum_{n=1}^{N} \frac{2}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{L} + \frac{1}{n\pi}\left(\left(-1\right)^{n+1}+1\right)\sin\frac{2n\pi x}{L}$$

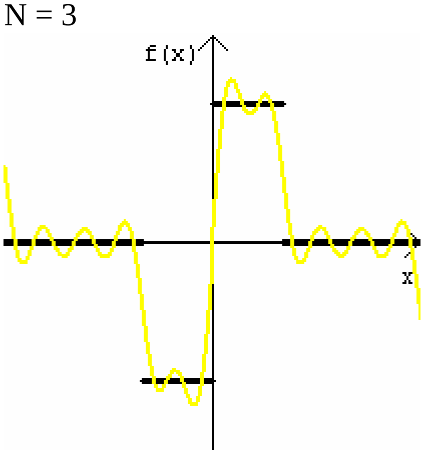
N = 1



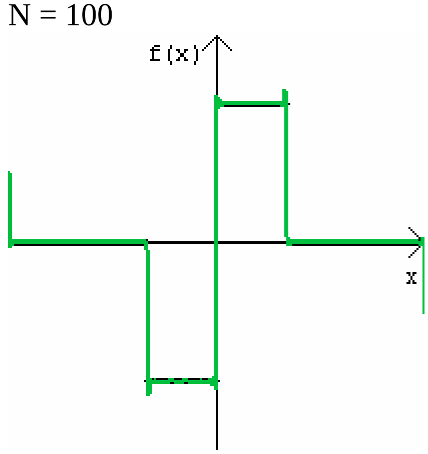
N = 2



N = 3

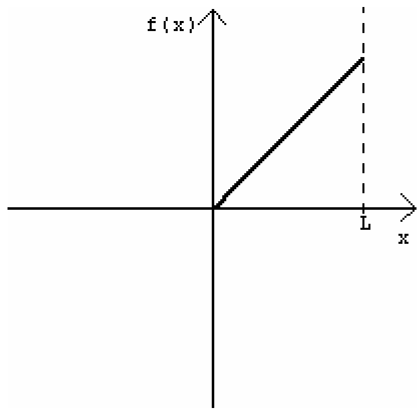


N = 100

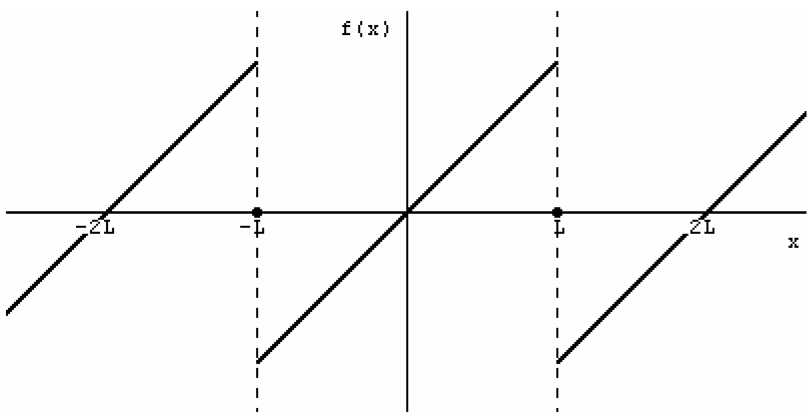


(c) $\boldsymbol{f(x) = x}$

The graph of the function *f*(*x*) is shown below for $0 \le x \le L$.



The Fourier sine series corresponds to the odd periodic extension. At discontinuities, the Fourier sine series takes on values halfway between the two values of the function. So the Fourier sine series for *f* looks like



The Fourier sine series has the form

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

Integrating by parts, we get

$$u = x \qquad dv = \sin\frac{n\pi x}{L}dx$$

$$du = dx \qquad v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\left[\left(-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx\right]$$

$$= \frac{2}{L}\left[\left(-\frac{L}{n\pi}L(-1)^n\right) + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L}\Bigg|_0^L\right]$$

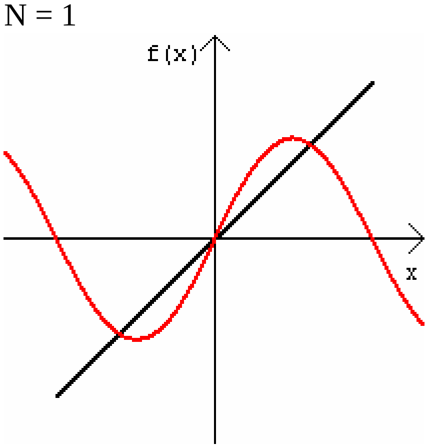
$$= \frac{2L}{n\pi}(-1)^{n+1}$$

So $f(x) = \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$

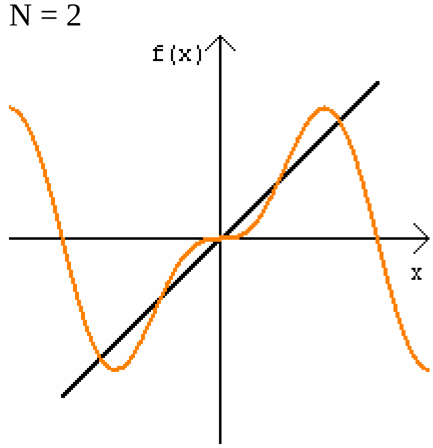
Below are some sketches of finite sums of the Fourier sine series

$$f(x) = \sum_{n=1}^{N}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$$

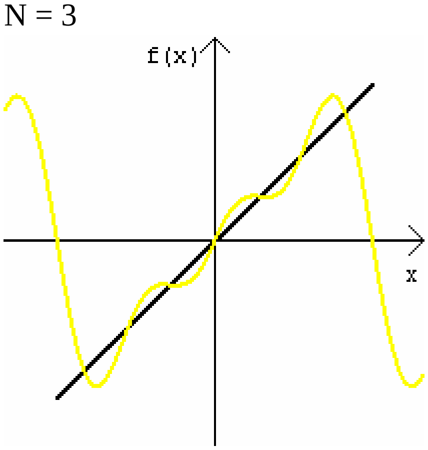
N = 1



N = 2



N = 3



N = 100