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Assignment 7 Extra Exercise
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(a) Show that convolution is commutative ($f_1 * f_2 = f_2 * f_1$).

The convolution of two functions is defined as

$$f_1 * f_2 \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(u)\, f_2(x-u)\, du$$

We can define a new variable $v = x - u$.

Defining all of the other variables in terms of *v*,

$$u = x - v$$

$$du = -dv$$

The limits of the integral also change. As $u \to -\infty$, $v \to \infty$ and as $u \to \infty$, $v \to -\infty$

Substituting these values into the problem, we get

$$f_1 * f_2 = -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f_1(x-v)\, f_2(v)\, dv$$

If we flip the limits of integration of the integral, the resulting minus sign cancels the minus sign that is already in the equation.

$$f_1 * f_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(v)\, f_1(x-v)\, dv$$

This equation has the form of convolution, but according to the definition

$$f_2 * f_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(u)\, f_1(x-u)\, du$$

Therefore,

$$f_1 * f_2 = f_2 * f_1$$

(b) Prove the convolution theorem (the formula for the inverse Fourier transform of a product of two functions of *k*).

We want to prove that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\hat{g}(k)\,dk = f(x)*g(x)$$

We define the Fourier transform and inverse Fourier transform as

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\,dx$$

So the product of two Fourier transforms is

$$\hat{f}(k)\hat{g}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx\int_{-\infty}^{\infty} e^{-ik\tilde{x}} g(\tilde{x})\,d\tilde{x}$$

We distinguish between the two *x*'s by using $\tilde{x}$ for the variable of integration for the Fourier transform of *g*. We can not combine the two integrals into one double integral.

$$\hat{f}(k)\hat{g}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ik(x+\tilde{x})} f(x)g(\tilde{x})\,dx d\tilde{x}$$

If we define $\tilde{x}=u-x$, $d\tilde{x}=du$

$$\hat{f}(k)\hat{g}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iku} f(x)g(u-x)\,dx du$$

Rearranging the equation a bit,

$$\hat{f}(k)\hat{g}(k)=\frac{1}{\sqrt{2\pi}}\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iku}\left(\int_{-\infty}^{\infty} f(x)g(u-x)\,dx\right)du}_{\text{Fourier transform of }\int_{-\infty}^{\infty} f(x)g(u-x)dx}$$

We can easily see the form of the Fourier transform in this equation. If we now take the inverse Fourier transform of the whole equation, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\hat{g}(k)\,dk=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)g(u-x)\,dx$$

$$= f(x)*g(x)$$