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Assignment 12 Exercise 7.10.12
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Solve Laplace's equation inside a hemisphere $\rho < a$ with $z > 0$ subject to $u = 0$ at $z = 0$ and the potential given on the hemisphere: $u(a,\theta,\phi) = F(\theta,\phi)$. [*Hint*: Use symmetry and solve a different problem, a sphere with the antisymmetric potential on the lower hemisphere.]

For this problem, we will use the book's convention; that is, the radial distance is ρ, the angle from the *z*-axis (colatitude) is ϕ, and the cylindrical angle (longitude) is θ.

The Laplacian in spherical coordinates is

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}$$

The problem we are to solve is

$$\nabla^2 u = 0 \qquad 0 \le \rho < a,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta < 2\pi$$

$$u\left(\rho,\theta,\frac{\pi}{2}\right) = 0$$

$$u(a,\theta,\phi) = F(\theta,\phi)$$

Solving the problem on the sphere with the antisymmetric potential on the lower hemisphere

$$\nabla^2 u = 0 \qquad 0 \le \rho < a,\ 0 \le \phi \le \pi,\ 0 \le \theta < 2\pi$$

$$u(a,\theta,\phi) = -u(a,-\theta,\pi-\phi) = F(\theta,\phi)$$

should give the same answer.

Standard separation of variables $u(\rho,\theta,\phi) = \mathrm{P}(\rho)\Theta(\theta)\Phi(\phi)$ gives

$$\frac{\left(\rho^2\mathrm{P}'\right)'}{\mathrm{P}} + \frac{1}{\sin\phi}\frac{\left(\sin\phi\,\Phi'\right)'}{\Phi} + \frac{1}{\sin^2\phi}\frac{\Theta''}{\Theta} = 0$$

We introduce a separation constant to get an ODE for ρ.

$$-\frac{\left(\rho^2 \mathrm{P}'\right)'}{\mathrm{P}} = -K = \frac{1}{\sin\phi}\frac{\left(\sin\phi\,\Phi'\right)'}{\Phi} + \frac{1}{\sin^2\phi}\frac{\Theta''}{\Theta}$$

$$-\frac{\left(\rho^2 \mathrm{P}'\right)'}{\mathrm{P}} = -K \qquad \sin\phi\frac{\left(\sin\phi\,\Phi'\right)'}{\Phi} + \frac{\Theta''}{\Theta} = -K\sin^2\phi$$

Separating again,

$$\sin\phi\frac{\left(\sin\phi\,\Phi'\right)'}{\Phi} + K\sin^2\phi = -\frac{\Theta''}{\Theta} = m^2$$

$$-\frac{\Theta''}{\Theta} = m^2 \qquad \frac{1}{\sin\phi}\left(\sin\phi\,\Phi'\right)' + \left[K - \frac{m^2}{\sin^2\phi}\right]\Phi = 0$$

The solution in θ is

$$\Theta(\theta) = A\cos(m\theta) + B\sin(m\theta)$$

We let $K = l(l+1)$, then the equations in ρ and ϕ become

$$\mathrm{P}'' + \frac{2}{\rho}\mathrm{P}' - \frac{l(l+1)}{\rho^2}\mathrm{P} = 0$$

$$\frac{1}{\sin\phi}\left(\sin\phi\,\Phi'\right)' + \left[l(l+1) - \frac{m^2}{\sin^2\phi}\right]\Phi = 0$$

The equation in ρ is a spherical Bessel function whose solutions are ρ^l and $\rho^{-(l+1)}$. We are considering only bounded solutions at $\rho = 0$, so we only use

$$\mathrm{P}(\rho) = \rho^l$$

The solution to the equation in ϕ is the Legendre function

$$\Phi(\phi) = P_l^m(\cos\phi)$$

So the solution of Laplace's equation in the sphere is

$$u(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{l=m}^{\infty}\rho^l\left[A_{lm}\cos(m\theta) + B_{lm}\sin(m\theta)\right]P_l^m(\cos\phi)$$

Applying the boundary condition, $u(a,\theta,\phi) = -u(a,-\theta,\pi-\phi) = F(\theta,\phi)$,

$$F(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{l=m}^{\infty} a^l \left[A_{lm}\cos(m\theta) + B_{lm}\sin(m\theta)\right] P_l^m(\cos\phi)$$

$$\begin{aligned} F(\theta,\phi) &= -\sum_{m=0}^{\infty}\sum_{l=m}^{\infty} a^l \left[A_{lm}\cos(-m\theta) + B_{lm}\sin(-m\theta)\right] P_l^m(\cos(\pi-\phi)) \\ &= -\sum_{m=0}^{\infty}\sum_{l=m}^{\infty} a^l \left[A_{lm}\cos(m\theta) - B_{lm}\sin(m\theta)\right] P_l^m(-\cos\phi) \end{aligned}$$

We see that these equations will only match if $B_{lm} = 0$ and if P_l^m is an odd function so that $P_l^m(-\cos\phi) = -P_l^m(\cos\phi)$. P_l^m is odd if l is odd so we let $l = 2n + 1$. Therefore,

$$u(\rho,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} A_{nm}\rho^{2n+1}\cos(m\theta) P_{2n+1}^m(\cos\phi)$$

$$A_{nm} = \frac{1}{a^{2n+1}} \frac{\int_0^{2\pi} d\theta \int_0^{\pi} F(\theta,\phi)\cos m\theta P_{2n+1}^m(\cos\phi)\sin\phi \, d\phi}{\int_0^{2\pi} d\theta \int_0^{\pi} \cos^2 m\theta \left[P_{2n+1}^m(\cos\phi)\right]^2 \sin\phi \, d\phi}$$