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Assignment 11 Exercise 7.5.2
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Consider the two-dimensional eigenvalue equation with $\sigma > 0$

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0$$

$$\phi = 0 \text{ on the boundary.}$$

(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal (with what weight?)

Since both eigenfunctions are zero on the boundary, then the line integral on the boundary must equal zero.

$$\oint(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1)\cdot\hat{n}\, ds = 0$$

By Green's second identity,

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \oint(u\nabla v - v\nabla u)\cdot\hat{n}\, ds$$

we can conclude that

$$\iint_R \left(\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1\right) dxdy = 0$$

The two eigenfunctions satisfy

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0$$

$$\nabla^2\phi = -\lambda\sigma(x, y)\phi$$

Substituting this expression into the equation,

$$\iint_R \left(\phi_1\left(-\lambda_2\sigma(x, y)\phi_2\right) - \phi_2\left(-\lambda_1\sigma(x, y)\phi_1\right)\right) dxdy = 0$$

$$(\lambda_1 - \lambda_2)\iint_R \sigma(x, y)\phi_1\phi_2\, dx\, dy = 0$$

$\lambda_1 \neq \lambda_2$ so $\lambda_1 - \lambda_2 \neq 0$, which means

$$\iint_R \sigma(x, y)\phi_1\phi_2\, dx\, dy = 0$$

So eigenfunctions belonging to different eigenvalues are orthogonal with weight $\sigma(x, y)$.

(b) Prove that all the eigenvalues are real.

Any real eigenvalue λ and corresponding eigenfunction ϕ satisfy the equation

$$\nabla^2\phi + \lambda\,\sigma\,\phi = 0$$

We let $\overline{\lambda}$ denote the complex conjugate of λ and $\overline{\phi}$ denote the eigenfunction corresponding to $\overline{\lambda}$. Then $\overline{\lambda}$ and $\overline{\phi}$ satisfy the equation

$$\nabla^2\overline{\phi} + \overline{\lambda}\,\sigma\,\overline{\phi} = 0$$

According to the result of part (a),

$$\left(\lambda - \overline{\lambda}\right)\iint_R \sigma\,\phi\,\overline{\phi}\,dx\,dy = 0$$

But since $\overline{\phi}$ is the complex conjugate of ϕ, we know that

$$\phi\,\overline{\phi} = \left|\phi\right|^2$$

$\left|\phi\right|^2 = 0$ only for $\phi \equiv 0$, which is not allowed since ϕ is an eigenfunction.

Also, we know that $\sigma > 0$, which means

$$\lambda - \overline{\lambda} = 0$$

$$\overline{\lambda} = \lambda$$

Since λ is real, all eigenvalues must be real.

(c) Prove that $\lambda \geq 0$.

Starting with the eigenvalue equation

$$\nabla^2\phi + \lambda\sigma\phi = 0$$

we multiply both sides by ϕ and integrate over the two-dimensional region

$$\iint_R \phi\,\nabla^2\phi\,dx\,dy + \lambda\iint_R \sigma\,\phi^2\,dx\,dy = 0$$

Solving for λ,

$$\lambda = \frac{-\iint_R \phi\,\nabla^2\phi\,dx\,dy}{\iint_R \sigma\,\phi^2\,dx\,dy}$$

From the product rule,

$$\nabla\cdot\left(\phi\,\nabla\phi\right) = \phi\,\nabla^2\phi + \left|\nabla\phi\right|^2$$

$$\phi\,\nabla^2\phi = \nabla\cdot\left(\phi\,\nabla\phi\right) - \left|\nabla\phi\right|^2$$

Substituting this expression into the integral,

$$\lambda = \frac{-\iint_R \nabla \cdot (\phi \nabla \phi)\, dx\, dy + \iint_R |\nabla \phi|^2\, dx\, dy}{\iint_R \sigma \phi^2 dx\, dy}$$

We know that

$$\iint_R \nabla \cdot \vec{A}\ dx\ dy = \oint \vec{A} \cdot \hat{n}\ ds$$

So

$$\iint_R \nabla \cdot (\phi \nabla \phi)\, dx\, dy = \oint \phi \nabla \phi \cdot \hat{n}\, ds$$

This term equals zero since the $\phi = 0$ on the boundary. Therefore,

$$\lambda = \frac{\iint_R |\nabla \phi|^2\, dx\, dy}{\iint_R \sigma \phi^2 dx\, dy}$$

$|\nabla \phi|^2 > 0$ and $\phi^2 > 0$ since they are squared terms. Also, $\sigma > 0$ is specified in the problem. Integrating positive functions always gives a positive value, so $\lambda \geq 0$.

(d) Is $\lambda = 0$ an eigenvalue, and if so, what is the eigenfunction?

If we take $\lambda = 0$, using the equation for λ found in part (c), the numerator must equal zero.

$$\iint_R |\nabla \phi|^2\, dx\, dy = 0$$

It follows that

$$|\nabla \phi|^2 = 0$$

$$\nabla \phi = \vec{0}$$

The gradient of ϕ is defined as

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)$$

Therefore,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$$

This means that ϕ is a constant function. We know that $\phi = 0$ on the boundary, so $\phi = 0$ everywhere. $\phi \equiv 0$ is not an eigenfunction, so $\lambda = 0$ is not an eigenvalue.