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Assignment 10 Exercise 7.8.7
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Consider Bessel's differential equation

$$\mathbf{z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + \left(z^2 - m^2\right) f = 0}$$

Let $f = y/z^{1/2}$. Derive that

$$\mathbf{\frac{d^2 y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0}$$

First, we substitute the definition of *f* into Bessel's equation

$$z^2 \frac{d^2}{dz^2}\left(yz^{-1/2}\right) + z\frac{d}{dz}\left(yz^{-1/2}\right) + \left(z^2 - m^2\right)\left(yz^{-1/2}\right) = 0$$

We assume $y = y(z)$ since $\frac{d^2 y}{dz^2}$ appears in the solution. Taking the derivatives of *f* in *z*,

$$\frac{d}{dz}\left(yz^{-1/2}\right) = y'z^{-1/2} - \frac{1}{2}yz^{-3/2}$$

$$\frac{d^2}{dz^2}\left(yz^{-1/2}\right) = y''z^{-1/2} - y'z^{-3/2} + \frac{3}{4}yz^{-5/2}$$

Substituting in these values, we get

$$z^2\left(y''z^{-1/2} - y'z^{-3/2} + \frac{3}{4}yz^{-5/2}\right) + z\left(y'z^{-1/2} - \frac{1}{2}yz^{-3/2}\right) + \left(z^2 - m^2\right)\left(yz^{-1/2}\right) = 0$$

$$y''z^{3/2} - \cancel{y'z^{1/2}} + \left(\frac{3}{4} - \frac{1}{2}\right)yz^{-1/2} + \cancel{y'z^{1/2}} + yz^{3/2} - m^2 yz^{-1/2} = 0$$

$$y''z^{3/2} + \frac{1}{4}yz^{-1/2} + yz^{3/2} - m^2 yz^{-1/2} = 0$$

Dividing the whole equation by $z^{3/2}$,

$$y'' + \frac{1}{4}yz^{-2} + y - m^2 yz^{-2} = 0$$

Rearranging the terms, we get the desired solution

$$\frac{d^2 y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0$$