Angela Sung arsung
Assignment 11 Exercise 7.8.8
http://people.tamu.edu/~arsung

Consider the solution to Bessel's equation $f = y/z^{1/2}$ that satisfies

$$\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0$$

Determine exact expressions for $J_{1/2}(z)$ and $Y_{1/2}(z)$. Use and verify the equations

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \quad \text{as } z \to \infty$$
$$Y_m(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \quad \text{as } z \to \infty \qquad \textbf{(7.8.3)}$$

$$J_m(z) \sim \frac{1}{2^m m!} z^m \quad \text{as } z \to 0$$
$$Y_m(z) \sim -\frac{2^m (m-1)!}{\pi} z^{-m} \quad \text{as } z \to 0 \qquad \textbf{(7.7.33)}$$

Since we are looking for $J_{1/2}(z)$ and $Y_{1/2}(z)$, we know that $m = \frac{1}{2}$.
The differential equation becomes

$$\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - \left(\frac{1}{2}\right)^2 z^{-2}\right) = 0$$

$$y''(z) + y(z) = 0$$

This tells us that *y* has the form

$$y(z) = A\cos z + B\sin z$$

and *f* has the form

$$f = z^{-1/2}\left[A\cos z + B\sin z\right]$$

The general solution to Bessel's equation is

$$f = c_1 J_m(z) + c_2 Y_m(z)$$

so we can define $J_{1/2}$ and $Y_{1/2}$ as

$$J_{1/2}(z) = z^{-1/2}\left[a\cos z + b\sin z\right]$$

$$Y_{1/2}(z) = z^{-1/2}\left[c\cos z + d\sin z\right]$$

As $z \to 0$, we get

$$J_{1/2}(z) \sim az^{-1/2}$$

However, from (7.7.33) we see that we want $J_{1/2}(z) \propto z^{1/2}$ for small z.
Therefore, we set $a = 0$, This gives us

$$J_{1/2}(z) = bz^{-1/2} \sin z$$

It is not readily apparent that this function is proportional to $z^{1/2}$ near $z = 0$, but we can use the approximation $\sin z \approx z$ for small z. Then

$$J_{1/2}(z) \sim bz^{1/2} \qquad \text{as } z \to 0$$

Using (7.7.33),

$$b = \frac{1}{2^{1/2}\,\frac{1}{2}!}$$

Now looking at the equation for $Y_{1/2}(z)$,

$$Y_{1/2}(z) \sim cz^{-1/2} \text{ as } z \to 0$$

which is consistent with (7.7.33) and we get

$$c = -\frac{2^{1/2}\left(-\frac{1}{2}\right)!}{\pi}$$

Non-integral factorials can be represented by the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

as

$$\Gamma(n+1) = n!$$

The gamma function has the properties

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Therefore,

$$\tfrac{1}{2}! = \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{\sqrt{\pi}}{2}$$

Similarly,

$$\left(-\tfrac{1}{2}\right)! = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Substituting these equations into the equations for *b* and *c*,

$$b = \frac{2}{2^{1/2}\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}$$

$$c = -\frac{2^{1/2}\sqrt{\pi}}{\pi} = -\sqrt{\frac{2}{\pi}}$$

Our solutions are now

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z$$

$$Y_{1/2}(z) = -\sqrt{\frac{2}{\pi z}}\cos z + dz^{-1/2}\sin z$$

We now compare our equations to those in (7.8.3):

As $z \to \infty$

$$J_{1/2}(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - \frac{1}{2}\frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi z}}\sin z$$

$$Y_{1/2}(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4} - \frac{1}{2}\frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{2}\right) = -\sqrt{\frac{2}{\pi z}}\cos z$$

We can see that our equation for $J_{1/2}(z)$ displays the correct behavior as $z \to \infty$. In order for $Y_{1/2}(z)$ to converge to $-\sqrt{\frac{2}{\pi z}}\cos z$ as $z \to \infty$, we need $d = 0$.

So we have our final solutions,

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z$$

$$Y_{1/2}(z) = -\sqrt{\frac{2}{\pi z}}\cos z$$