Angela Sung arsung
Assignment 8 Problem 2
http://people.tamu.edu/~arsung

Solve the wave equation in $\mathbb{R}^2$ (one space variable $-\infty < x < \infty$, one time variable $-\infty < t < \infty$, $c = 1$) by Fourier transforms. Rearrange the result into the form

$$u(x,t) = \int_{-\infty}^{\infty} dy \left[W(x-y,t) f(y) + V(x-y,t) g(y) \right]$$

where *f* and *g* are the initial data and *W* and *V* are certain integral expressions. What identities for the Fourier transforms of delta functions and step functions do you need to assume in order to get your answer to agree with the one on p. 80 of the notes? Try to boil it down to simple formulas for functions (or distributions) of *one* variable, not three.

The wave equation in $\mathbb{R}^2$ is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty,\ -\infty < t < \infty$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

We will use the Fourier transform definitions

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\,dk$$

Taking the Fourier transform in *x*,

$$\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = -k^2 \hat{u}(k,t)$$

$$\hat{u}(k,0) = \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \qquad \frac{\partial \hat{u}}{\partial t}(k,0) = \hat{g}(k) = \int_{-\infty}^{\infty} e^{-ikx} g(x)\,dx$$

The general solution to the PDE is

$$\hat{u}(k,t) = A(k)\cos(kt) + B(k)\sin(kt)$$

Applying the initial conditions

$$\hat{u}(k,0) = \hat{f}(k) = A(k)$$

$$\frac{\partial \hat{u}}{\partial t}(k,0) = \hat{g}(k) = kB(k) \Rightarrow B(k) = \frac{\hat{g}(k)}{k}$$

So,

$$\hat{u}(k,t) = \hat{f}(k)\cos(kt) + \frac{\hat{g}(k)}{k}\sin(kt)$$

Now we take the inverse Fourier transform to recover the original function.

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\hat{u}(t,k)\,dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\left[\hat{f}(k)\cos(kt)+\frac{\hat{g}(k)}{k}\sin(kt)\right]dk$$

Substituting the equations for $\hat{f}(k)$ and $\hat{g}(k)$ into our equation for $u(x,t)$

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\left[\left(\int_{-\infty}^{\infty}e^{-iky}f(y)\,dy\right)\cos(kt)+\left(\int_{-\infty}^{\infty}e^{-iky}g(y)\,dy\right)\frac{\sin(kt)}{k}\right]dk$$

Rearranging a bit to get the desired form

$$u(x,t)=\int_{-\infty}^{\infty}dy\left[W(x-y,t)\,f(y)+V(x-y,t)\,g(y)\right]$$

$$W(x-y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-y)}\cos(kt)\,dk \qquad V(x-y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x-y)}\sin(kt)}{k}\,dk$$

p. 80 of the notes says

$$u(x,t)=\int_{-\infty}^{\infty}\frac{\partial G}{\partial t}(x,y,t)\,u(y,0)\,dy+\int_{-\infty}^{\infty}G(x,y,t)\frac{\partial u}{\partial t}(y,0)\,dy$$

$$G(x,y,t)=\frac{1}{2}\left[h(y-x+t)-h(y-x-t)\right]$$

$$\frac{\partial G}{\partial t}(x,y,t)=\frac{1}{2}\left[\delta(y-x+t)+\delta(y-x-t)\right]$$

where h is the unit step function and δ is the Dirac delta function.

We can write G in terms of two variables to match W and V.

$$G(y-x,t)=\frac{1}{2}\left[h(y-x+t)-h(y-x-t)\right] \qquad \frac{\partial G}{\partial t}(y-x,t)=\frac{1}{2}\left[\delta(y-x+t)+\delta(y-x-t)\right]$$

so that

$$G(z,t)=\frac{1}{2}\left[h(z+t)-h(z-t)\right] \qquad \frac{\partial G}{\partial t}(z,t)=\frac{1}{2}\left[\delta(z+t)+\delta(z-t)\right]$$

Similarly for W and V

$$W(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikz}\cos(kt)\,dk \qquad V(z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikz}\sin(kt)}{k}\,dk$$

In order for the solutions to be the same,

$$W(z,t)=\frac{\partial G}{\partial t}(z,t) \quad \Rightarrow \quad \delta(z+t)+\delta(z-t)=\frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikz}\cos(kt)\,dk \tag{1}$$

$$V(z,t)=G(z,t) \quad \Rightarrow \quad h(z+t)-h(z-t)=\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikz}\sin(kt)}{k}\,dk \tag{2}$$

Euler's formula says

$$\cos(t)=\frac{e^{it}+e^{-it}}{2} \qquad \sin(t)=\frac{e^{it}-e^{-it}}{2i}$$

so we can rewrite (1) and (2) as

$$\delta(z+t)+\delta(z-t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{ik(z+t)}+e^{ik(z-t)}\right)dk$$

$$h(z+t)-h(z-t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ik(z+t)}-e^{ik(z-t)}}{k}\,dk$$

From these two equations, we can see that

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk \qquad h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{ik}\,dk$$

For any horizontal translation of δ or h,

$$\delta(x-a)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-a)}\,dk \qquad h(x-a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(x-a)}}{ik}\,dk$$

The right sides have the form of inverse Fourier transforms

$$\delta(x-a)=\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-ika}\,dk}_{\substack{\text{inverse Fourier}\\ \text{transform of } e^{-ika}}} \qquad h(x-a)=\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{e^{-ika}}{ik}\,dk}_{\substack{\text{inverse Fourier}\\ \text{transform of } \frac{e^{-ika}}{ik}}}$$

So the Fourier transforms of the delta and step functions must be

$$\int_{-\infty}^{\infty} e^{-ikx}\delta(x-a)\,dx=e^{-ika}$$

$$\int_{-\infty}^{\infty} e^{-ikx}h(x-a)\,dx=\frac{e^{-ika}}{ik}$$