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Assignment 7 Exercise 9.3.22
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Solve

$$\frac{dG}{dx}+G=\delta(x-x_0) \qquad \text{with } G(0,x_0)=0.$$

Show that $G(x,x_0)$ is not symmetric even though $\delta(x-x_0)$ is and use the result to solve

$$y'(x)+y(x)=f(x) \qquad y(0)=0$$

for arbitrary f.

The equation would be much easier to solve without the delta function. In order to satisfy the delta function and remove it from the equation, we can integrate over a small interval $[x_0-\varepsilon, x_0+\varepsilon]$.

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon}\frac{dG}{dx}dx=\int_{x_0-\varepsilon}^{x_0+\varepsilon}\delta(x-x_0)-Gdx$$

$\int_{x_0-\varepsilon}^{x_0+\varepsilon}Gdx\to 0$ as $\varepsilon\to 0$, so the integral evaluates as

$$G(x,x_0)\Big|_{x_0-\varepsilon}^{x_0+\varepsilon}=G(x_0^+,x_0)-G(x_0^-,x_0)=1$$

and we are left with the homogeneous equation

$$\frac{dG}{dx}+G=0 \qquad \text{for } x\neq x_0.$$

This equation gives us

$$G(x,x_0)=A(x_0)e^{-x} \qquad x<x_0$$

$$G(x,x_0)=C(x_0)e^{-x} \qquad x>x_0$$

Applying the boundary conditions,

$$G(0,x_0)=0\Rightarrow A(x_0)=0$$

$$G(x_0^+,x_0)-G(x_0^-,x_0)=1$$

$$C(x_0)e^{-x_0}=1$$

$$C(x_0)=e^{x_0}$$

So we have $G(x,x_0)=\begin{cases}0 & x<x_0\\ e^{x_0}e^{-x} & x>x_0\end{cases}$

$G(x,x_0)$ is noticeably not symmetric about x_0 as seen from the above equation. It has some function value for $x>x_0$, but is constantly zero for $x<x_0$.

We now need to solve

$$y'(x) + y(x) = f(x) \qquad\qquad y(0) = 0$$

We know that the differential equation can be solved by some Green function

$$y(x) = \int_0^\infty G(x, x_0)\, f(x_0)\, dx_0$$

The nonhomogeneous term the differential equation can also be expressed as

$$f(x) = \int_0^\infty \delta(x - x_0)\, f(x_0)\, dx_0$$

If we plug these equations into the original differential equation, the result is

$$\left(\frac{d}{dx} + 1\right)\int_0^\infty G(x, x_0)\, f(x_0)\, dx_0 = \int_0^\infty \delta(x - x_0)\, f(x_0)\, dx_0$$

Since the differential operator is linear, we can move it inside the integral

$$\int_0^\infty \left(\frac{d}{dx} + 1\right) G(x, x_0)\, f(x_0)\, dx_0 = \int_0^\infty \delta(x - x_0)\, f(x_0)\, dx_0$$

And so we can see that $y(x)$ should be defined as

$$y(x) = \int_0^\infty G(x - x_0)\, f(x_0)\, dx_0$$

to match the delta function. This tells us that the differential equation can be solved by solving

$$\left(\frac{d}{dx} + 1\right) G = \delta(x - x_0)$$

This is the same equation that was solved in the first part. In addition, the solution to the first part uses the boundary condition $G(0, x_0) = 0$, which corresponds to the boundary condition for the differential equation, $y(0) = 0$.

So the Green function found in the first part solves the differential equation. Therefore, the solution to the differential equation is

$$y(x) = \int_0^\infty G(x - x_0)\, f(x_0)\, dx_0$$

$$G(x, x_0) = \begin{cases} 0 & x < x_0 \\ e^{x_0} e^{-x} & x > x_0 \end{cases}$$