Consider the following wave equation for a vibrating rectangular membrane

$$0 < x < L,\ 0 < y < H$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

With the initial conditions:

$$u(x,y,0) = 0$$

$$\frac{\partial u}{\partial t}(x,y,0) = f(x,y)$$

and boundary conditions:

$$\frac{\partial u}{\partial x}(0,y,t) = 0$$

$$\frac{\partial u}{\partial x}(L,y,t) = 0$$

$$\frac{\partial u}{\partial y}(x,0,t) = 0$$

$$\frac{\partial u}{\partial y}(x,H,t) = 0$$

First, we separate only the time variable,

$$u(x,y,t) = \varphi(x,y)h(t)$$

Plug this into the original equation

$$\frac{d^2 h}{dt^2}\varphi = c^2 h\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right)$$

then separate h and φ and set both sides equal to a constant, $-\lambda$

$$\frac{1}{hc^2}\frac{d^2 h}{dt^2} = \frac{1}{\varphi}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = -\lambda$$

yielding two eigenvalue problems:

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -\lambda\varphi$$

For now, we will focus on the φ problem, and separate it further into Cartesian coordinates.

$$\varphi(x,y) = f(x)g(y)$$

We plug this into the eigenvalue problem

$$g\frac{d^2 f}{dx^2} + f\frac{d^2 g}{dy^2} = -\lambda fg$$

then separate f and g and set both sides to a constant, $-\mu$

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2 g}{dy^2} = -\mu$$

yielding two more eigenvalue problems:

$$\frac{d^2 f}{dx^2} = -\mu f$$

$$\frac{d^2 g}{dy^2} = -(\lambda - \mu) g$$

We start by solving for f(x), since it only involves one unknown constant and has two homogeneous boundary conditions.

$$\frac{d^2 f}{dx^2} = -\mu f \text{ with } \frac{df}{dx}(0) = 0 \text{ and } \frac{df}{dx}(L) = 0$$

$$f(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

$$f'(x) = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x = 0$$

$$f'(0) = \sqrt{\lambda}c_2 = 0$$

$$c_2 = 0$$

$$f'(L) = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}L = 0$$

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3...$$

$$f_n(x) = \cos\frac{n\pi x}{L}$$

Now we use this information to solve g(y)

$$\frac{d^2 g}{dy^2} = -(\lambda - \mu) g \text{ with } \frac{dg}{dy}(0) = 0 \text{ and } \frac{dg}{dy}(H) = 0$$

$$g(y) = d_1 \cos\sqrt{\lambda - \mu}y + d_2 \sin\sqrt{\lambda - \mu}y$$

$$g'(y) = -\sqrt{\lambda - \mu}d_1 \sin\sqrt{\lambda - \mu}y + \sqrt{\lambda - \mu}d_2 \cos\sqrt{\lambda - \mu}y$$

$$g'(0) = \sqrt{\lambda - \mu}d_2 = 0$$

$$d_2 = 0$$

$$g'(H) = -\sqrt{\lambda - \mu}d_1 \sin\sqrt{\lambda - \mu}H = 0$$

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2 \qquad m = 1, 2, 3...$$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$$g_{nm}(y) = \cos\frac{m\pi y}{H}$$

Now we attack the t part of the equation, knowing that all values of λ_{nm} must be positive, as it is the sum of two squares.

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \text{ with } h(0) = 0$$

$$h(t) = b_1 \cos c\sqrt{\lambda}t + b_2 \sin c\sqrt{\lambda}t$$

$$h(0) = b_1 = 0$$

$$h_{nm}(t)\begin{cases} t & m = n = 0 \\ \sin\left(ct\sqrt{\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\right) & otherwise \end{cases}$$

Finally, we combine the three parts to get formula for u(t).

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{nm} \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H} h_{nm}(t)$$

for

$$h_{nm}(t)\begin{cases} t & m = n = 0 \\ \sin\left(ct\sqrt{\left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\right) & otherwise \end{cases}$$

We then use the non homogeneous initial condition to solve for A_{nm}.

$$\frac{\partial u}{\partial t}(x,y,0) = f(x,y)$$

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{nm} \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H} h'_{nm}(0)$$

which is equivalent to

$$f(x,y) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} A_{nm} \cos\frac{n\pi x}{L}\right)\cos\frac{m\pi y}{H} h'_{nm}(0)$$

Using Fourier cosine series' in x and y, we obtain

$$A_{nm}h'_{nm}(0) = \frac{\int_0^H\int_0^L f \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}dxdy}{\int_0^H\int_0^L \cos^2\frac{n\pi x}{L}\cos^2\frac{m\pi y}{H}dxdy}$$