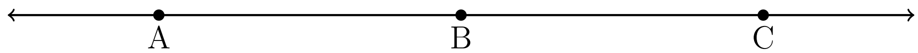
Proposition 3.6. Given $A * B * C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.



Proof. <u>Prove $\overrightarrow{AB} = \overrightarrow{AC}$.</u>

First we shall prove that $\overrightarrow{AB} \subseteq \overrightarrow{AC}$. By the definition of ray, this is equivalent to proving that

$$AB \cup \{P | A * B * P\} \subseteq AC \cup \{P | A * C * P\}$$

Let $P \in \overrightarrow{AB} = AB \cup \{P | A * B * P\}$ Then we have two cases:

- **Case 1:** Suppose $P \in AB$. Then $P \in AC$ by Proposition 3.5, since $A * B * C$.
- **Case 2:** Suppose $P \in \{P | A * B * P\}$. Then we have three cases:
  - Case a: Suppose $A * P * C$. Then $P \in AC$, so $P \in \overrightarrow{AC}$.
  - Case b: Suppose $P = C$. Then $P \in AC$, so $P \in \overrightarrow{AC}$.
  - Case a: Suppose $A * C * P$. Then $P \in \overrightarrow{AC}$ by definition.

Thus, $\overrightarrow{AB} \subseteq \overrightarrow{AC}$, and we shall now prove the converse direction, equivalent to

$$AC \cup \{P | A * C * P\} \subseteq AB \cup \{P | A * B * P\}$$

. Let $Q \in \overrightarrow{AC} = AC \cup \{Q | A * C * Q\}$. Then we have two cases:

- **Case 1:** Suppose $Q \in AC$. Then by Proposition 3.5, we have two cases, since $A * B * C$:
  - Case a: Suppose $Q \in AB$. Then $Q \in \overrightarrow{AB}$.
  - Case b: Suppose $Q \in BC$ and that $Q \neq B$. Then either $B * QC$ or $Q = C$. In either case, $A * B * Q$ by Proposition 3.5, so $Q \in \overrightarrow{AC}$.
- **Case 2:** Suppose $Q \in \{Q | A * C * Q\}$. Then $A * B * Q$ by Proposition 3.5, so $Q \in \overrightarrow{AC}$.

Thus, $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{AC}}$.

$\square$

Adapted from Group Delta, S2019