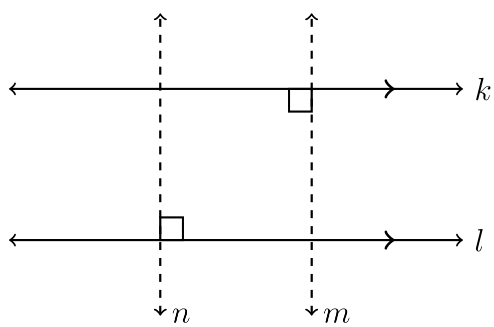
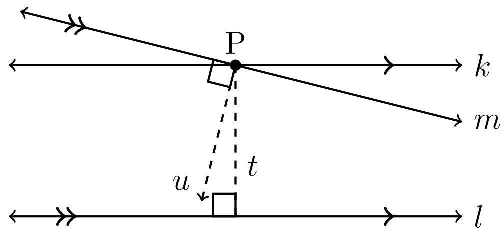
Proposition 4.10. Hilbert's Euclidean parallel postulate $\Leftrightarrow$ if $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$.

Proof. For the forward direction, take HE as hypothesis. Let $k \parallel l$, $m \perp k$, and $n \perp l$. The case where $m = n$ being trivial, suppose $m \neq n$. By Proposition 4.9, $n \perp k$, so by Corollary 1 of AIA, we have that $m \parallel n$, since k is perpendicular to both m and n.



For the converse direction, take as hypothesis that if $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n$. Let l be a line with point P not on l, let k be a parallel to l through P, and let t be the perpendicular to l through P. Suppose there exists a second parallel m to l through P, and let u be the perpendicular to m through P. Then by the proposition of our hypothesis, $u = t$ or $u \parallel t$. Since u and t are both incident to P, we may rule out the second case and conclude that $u = t$, or $m \perp t$. Now let s be the perpendicular to k through P. By a second application of the proposition, we conclude that $s = t$, or $k \perp t$. The lines k and m are both perpendicular to t at point P, meaning $k = m$ by the uniqueness component of C-4. Thus, k is the only line perpendicular to l through P, yielding HE. □

Adapted from Groups Alpha and Epsilon, F2010 and Group Theta, S2019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