Math. 311 (Fulling) 28 October 1996

Test B – Solutions

Name: ______________________________ Number: ______________

(as on attendance sheets)

1. *(16 pts.)* Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

   (a) $\{\sin(4t), e^{2t}, \sinh(2t), \sin(2t)\cos(2t)\}$

   Because $\sin(4t) = 2\sin(2t)\cos(2t)$, the set is not independent. It can be made independent by discarding either the first or the last element in the list. (Note that although e^{2t} and $\sinh(2t)$ are "related", they are not dependent; they would be dependent if either e^{-2t} or $\cosh(2t)$ were also in the list.)

   (b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

   Write the vectors as rows and reduce:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the most natural of many correct answers is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\} \qquad \text{(or} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\},$$

if you prefer to carry the reduction to completion).

2. *(15 pts.)* $L\colon \mathbf{R}^3 \to \mathbf{R}^2$ is represented by the matrix $\begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & 2 \end{pmatrix}$.

   (a) Find the kernel of L .

   Solve the homogeneous equation:

$$\begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \end{pmatrix};$$

$$x = 5z, \quad y = -z, \quad z \text{ arbitrary.}$$

(We have omitted the column of zeros at the right of the augmented matrix, since they will always stay zeros.) Thus the kernel is the span of the vector $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$, or all vectors of the form $\begin{pmatrix} 5z \\ -z \\ z \end{pmatrix}$ with $z \in \mathbf{R}$.

   (b) What is the rank of L ?

$$\dim \operatorname{ran} L = \dim \operatorname{dom} L - \dim \ker L = 3 - 1 = 2.$$

Alternatively, it is obvious that there are two independent columns. (The number of independent columns can't be 3, since they are only 2-component vectors. If the number of independent columns were 1 or 0, that would be very obvious.)

3. *(24 pts.)* Define $L: \mathcal{P}_3 \to \mathcal{P}_2$ by $L(p) = p'' - p'$.

(a) Find the matrix representing L with respect to the bases $\{t^3, t^2, t, 1\}$ for $\mathcal{P}_3$ and $\{t^2, t, 1\}$ for $\mathcal{P}_2$.

Calculate the action of L on each basis vector:

$$L(t^3) = 6t - 3t^2, \quad L(t^2) = 2 - 2t, \quad L(t) = -1, \quad L(1) = 0.$$

Translate each of these formulas into a column (with the top row corresponding to t^2 , etc.):

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix}.$$

(b) Is L surjective (onto)? If not, what is its range?

YES — the first three columns are independent and therefore span $\mathbf{R}^3$, so the corresponding images of basis vectors ($\{L(t^3), L(t^2), L(t)\}$) span $\mathcal{P}_2$.

(c) Is L injective (one-to-one)? If not, what is its kernel?

NO — $\dim\ker L = \dim\operatorname{dom} L - \dim\operatorname{ran} L = 4 - 3 = 1 \neq 0$. To find exactly what the kernel is: Row reduction yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(As in Qu. 2, this matrix has not been "augmented" by a useless column of zeros.) Therefore, if we represent a generic element of the domain as $p(t) = at^3 + bt^2 + ct + d$, the solution of the homogeneous equation $L(p) = 0$ is

$$a = b = c = 0, \qquad d \text{ arbitrary.}$$

Thus $\ker L$ consists of the constant functions, or $\ker L = \mathcal{P}_0$.

Alternatively, solve the homogeneous differential equation $p'' - p' = 0$: Let $v \equiv p'$, so $v' = v$. Thus $v = Ce^t$, and hence $p = Ce^t + D$, for arbitrary constants C and D . But these functions are not in the domain $\mathcal{P}_3$ (indeed, are not polynomials at all) unless $C = 0$. So again we conclude that the kernel is the constant functions.

4. *(10 pts.; bonus for very good answer)* Do **ONE** of these [(A) or (B)].

(A) Prove that the range of a linear function is a subspace of the function's codomain.

[Please reread the proof of Theorem 1 in Sec. 5.2.]

(B) Explain how one or more of the "superposition principles" is used in solving an inhomogeneous differential equation such as $y'' + 4y = \cos(3t)$.

[Please reread the Important Remarks in Sec. 5.2 and the Superposition Principles in Sec. 5.3, especially Principles 2 and 3.]

5. *(15 pts.)* Let $\mathcal{M}$ be the vector space of 3×3 matrices, and let $\mathcal{S}$ be the subset of such matrices whose elements satisfy $A_{31} = A_{12} + A_{23}$.

An example of a member of $\mathcal{S}$ is $\begin{pmatrix} 100 & 3 & 0 \\ -13 & -21 & 4 \\ 7 & -1 & 37 \end{pmatrix}$.

(a) Prove that $\mathcal{S}$ is a subspace of $\mathcal{M}$.

Quick argument: The equation $A_{31} = A_{12} + A_{23}$ is linear and homogeneous, so its solutions form a subspace.

More detailed argument: Let A and B be two elements of $\mathcal{S}$, and let $r \in \mathbf{R}$. Then

$$\begin{aligned}(rA+B)_{31} = rA_{31} + B_{31} &= r(A_{12}+A_{23}) + (B_{12}+B_{23}) \\ &= rA_{12} + B_{12} + rA_{23} + B_{23} = (rA+B)_{12} + (rA+B)_{23}\,.\end{aligned}$$

This shows that $\mathcal{S}$ is closed under addition and scalar multiplication (and it is obviously nonempty).

(b) What is the dimension of $\mathcal{S}$?

Eight. (The condition $A_{31} = A_{12} + A_{23}$ determines one of the 9 matrix elements in terms of the others. The remaining 8 are free to vary arbitrarily.)

6. *(20 pts.)* A linear function $L\colon \mathbf{R}^3 \to \mathbf{R}^2$ has the matrix $M = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix}$ with respect to the natural bases.

(a) Find the matrix of L with respect to the natural basis ($\{\hat e_1, \hat e_2, \hat e_3\}$) for $\mathbf{R}^3$ (the domain) and the basis $\left\{ \vec v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \vec v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ for $\mathbf{R}^2$ (the codomain).

Putting the vectors together, we get a matrix $U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, which maps the new coordinates into the natural coordinates. Thus $U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ maps the natural coordinates to the new coordinates. Therefore, the desired matrix is

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 0 & 5 \\ 1 & 0 & 1 \end{pmatrix}.$$

(b) Find the matrix of L with respect to the basis

$$\{\vec u_1 = \hat e_1 - 2\hat e_2,\ \vec u_2 = \hat e_1 + \hat e_2 + \hat e_3,\ \vec u_3 = \hat e_1 + 2\hat e_3\}$$

for $\mathbf{R}^3$ and the natural basis for $\mathbf{R}^2$.

Method 1: As in (a), the matrix with the $\vec u$ s as columns, which is $\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, maps the new coordinates to the natural ones. Therefore, the desired matrix is

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 5 & 8 \end{pmatrix}.$$

Method 2: From the formulas for the new basis in terms of the natural basis, we have that the matrix $K = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ maps the natural basis into the new basis. Therefore, its transpose maps the new coordinates to the natural coordinates. But $K^{\mathrm{t}} = U$, so the rest of the calculation proceeds as in the first method.

Method 3: Apply the " kth column rule" to the elements of the new basis:

$$L(\vec{u}_1) = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

similarly,

$$L(\vec{u}_2) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \qquad L(\vec{u}_3) = \begin{pmatrix} 5 \\ 8 \end{pmatrix}.$$

Put these together as the columns of the new matrix.