Math. 311 (Fulling) 25 November 1996

Test C – Solutions

Name: ______________________________ Number: ______________

(as on attendance sheets)

1. *(13 pts.)* Let $\vec{v}_1 = (1,1,1)$, $\vec{v}_2 = (1,0,2)$, $\vec{v}_3 = (1,-1,0)$.

(a) Find the volume of the parallelepiped with these vectors as edges.

The determinant of the three vectors is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} = 3.$$

(At the intermediate step I subtracted the first row from the other two to demonstrate the simplification of a determinant by row operations, although that was hardly necessary in such a simple case.) This is the volume (since it is already positive).

Equivalently, take the cross product of two of the vectors, then the dot product of the result with the third vector, and take the absolute value if necessary.

(b) Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a right-handed or a left-handed basis for $\mathbf{R}^3$?

Right-handed, since the determinant is positive.

2. *(12 pts.)* Prove the identity $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$.

Quick method: By the product rule, the object on the left consists of two classes of terms, those that involve derivatives of $\vec{A}$ and those that involve derivatives of $\vec{B}$. We can get the correct algebraic structure for each sort of term by applying the vector triple product identity, interpreting the ∇ as acting only on the functions to its right. Since $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times \vec{v})$, the $\vec{A}$ -derivative terms are $\vec{B} \cdot (\nabla \times \vec{A})$. Since $\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = -\vec{v} \cdot (\vec{u} \times \vec{w})$, the $\vec{B}$ -derivative terms are $-\vec{A} \cdot (\nabla \times \vec{B})$.

More careful, elementary method:

$$\begin{aligned} \nabla \cdot (\vec{A} \times \vec{B}) &= \frac{\partial}{\partial x}(A_y B_z - A_z B_y) + \frac{\partial}{\partial y}(A_z B_x - A_x B_z) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x) \\ &= \frac{\partial A_y}{\partial x} B_z + A_y \frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x} B_y - A_z \frac{\partial B_y}{\partial x} \\ &\qquad + \text{analogous terms with } y \text{ and } z \text{ derivatives.} \end{aligned}$$

Since it is clear that the three coordinates enter in a symmetrical way, we can concentrate on the x derivatives and be confident that the others will do the same thing. The x derivatives in $\vec{B} \cdot (\nabla \times \vec{A})$ are

$$B_x(0) - B_y \frac{\partial A_z}{\partial x} + B_z \frac{\partial A_y}{\partial x} .$$

Those in $-\vec{A} \cdot (\nabla \times \vec{B})$ are

$$-A_x(0) + A_y \frac{\partial B_z}{\partial x} - A_z \frac{\partial B_y}{\partial x} .$$

We observe that these add up to the right thing.

(Still more careful and elementary is to write *everything* out, but I don't have the patience to type all that.)

3. *(20 pts.)* Define a new coordinate system (u,v) in $\mathbf{R}^2$ by $x = u - e^v$, $y = v + e^u$.

(a) Find (as functions of u and v) the tangent vectors to the coordinate curves and the normal vectors to the coordinate "surfaces" (which are curves in this dimension).

The tangent vector to the curve $v =$ const. is

$$\frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix} = \begin{pmatrix} 1 \\ e^u \end{pmatrix}.$$

The tangent vector to the curve $u =$ const. is

$$\frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} -e^v \\ 1 \end{pmatrix}.$$

These are the columns of the Jacobian matrix,

$$\frac{d(x,y)}{d(u,v)} = \begin{pmatrix} 1 & -e^v \\ e^u & 1 \end{pmatrix},$$

whose determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 + e^{u+v}.$$

The normal vectors are the rows of the inverse of this matrix,

$$\frac{d(u,v)}{d(x,y)} = \frac{1}{1+e^{u+v}} \begin{pmatrix} 1 & e^v \\ -e^u & 1 \end{pmatrix}.$$

Namely,

$$\nabla u = \left(\frac{1}{1+e^{u+v}}, \frac{e^v}{1+e^{u+v}} \right)$$

is normal to the curve of constant u , and

$$\nabla v = \left(\frac{-e^u}{1+e^{u+v}}, \frac{1}{1+e^{u+v}} \right)$$

is normal to the curve of constant v .

(b) Calculate the area of the region bounded by the curves

$$u = 0, \quad v = 0, \quad u = 1, \quad v = \ln 2 \approx 0.69.$$

Integrate the Jacobian determinant (found in (a)) over the region with respect to u and v :

$$\int_{v=0}^{\ln 2} \int_{u=0}^{1} (1 + e^u e^v)\, du\, dv = \int_{v=0}^{\ln 2} (u + e^u e^v)\Big|_0^1 \, dv$$
$$= \int_{v=0}^{\ln 2} (1 + ee^v - e^v)\, dv = \big[v + (e-1)e^v\big]_0^{\ln 2}$$
$$= \ln 2 + 2(e-1) - (e-1) = \ln 2 + e - 1.$$

(c) *(extra credit)* In the x - y plane. sketch (and label) the vectors in (a) at the points $(u,v)=(0,0)$ and $(u,v)=(0,\ln 2)$, and sketch the region in (b).

First locate the points:

$$u=0,\ v=0\ \Rightarrow\ x=-1,\ y=1;$$
$$u=0,\ v=\ln 2\ \Rightarrow\ x=-2,\ y=1+\ln 2\approx 1.69.$$

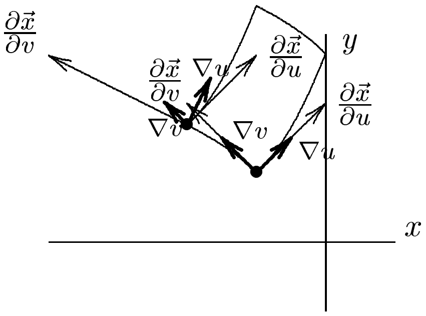
At $(u,v)=(0,0)$, the Jacobian matrices are

$$\frac{d(x,y)}{d(u,v)}=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},\qquad \frac{d(u,v)}{d(x,y)}=\begin{pmatrix}\frac12 & \frac12\\ -\frac12 & \frac12\end{pmatrix}.$$

Plot the columns of the first of these and the rows of the second with their tails at the point $(x,y)=(-1,1)$. At $(u,v)=(0,\ln 2)$, the Jacobian matrices are

$$\frac{d(x,y)}{d(u,v)}=\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix},\qquad \frac{d(u,v)}{d(x,y)}=\begin{pmatrix}\frac13 & \frac23\\ -\frac13 & \frac13\end{pmatrix}.$$

Plot the respective columns and rows with their tails at the point $(x,y)=(-2,1.69)$. To get the region, I plotted the other two corners and, to get the curvature on the sides right, also four midpoints. (Of course, in class nobody did all this, nor was expected to. But the final plot does look very much like the freehand sketch I made when composing the test.)



4. *(20 pts.)*

(a) Construct an orthonormal basis for the span of $\{\vec v_1=(1,1,1),\ \vec v_2=(1,0,2)\}$.

Normalize the first vector:

$$\hat u_1=\vec v_1/\|\vec v_1\|=\frac{1}{\sqrt3}(1,1,1).$$

Project the second vector onto the first: $\vec v_{\|}=\langle\vec v_2,\hat u_1\rangle\hat u_1=\frac33(1,1,1)=(1,1,1)$. Then $\vec v_\perp=\vec v_2-\vec v_{\|}=(0,-1,1)$. Normalize that vector:

$$\hat u_2=\frac{1}{\sqrt2}(0,-1,1).$$

Then $\{\hat u_1,\hat u_2\}$ is a basis for the span of the two given vectors (a 2-dimensional subspace of $\mathbf{R}^3$.)

(b) Find the expansion coefficients of $\vec{w} = (3, 2, 4)$ with respect to that orthonormal basis.

$$c_1 = \langle \vec{w}, \hat{u}_1 \rangle = \frac{9}{\sqrt{3}} = 3\sqrt{3}; \qquad c_2 = \langle \vec{w}, \hat{u}_2 \rangle = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Then $\vec{w} = c_1\hat{u}_1 + c_2\hat{u}_2$ (since $\vec{w}$ is in the span of the two given vectors).

5. *(10 pts.)* Do **ONE** of these [(A) or (B)].

(A) State the definition of an *inner product* on a vector space $\mathcal{V}$.

[See textbook.]

(B) Suppose that M is a square matrix and that all its elements are integers (as in most examples in this course). Suppose that $\det M = \pm 1$. Prove that M^{-1} also has only integer elements. HINT: Appeal to Cramer's rule.

Cramer's rule gives each element of the inverse matrix in the form

$$\frac{\text{cofactor}}{\text{determinant}}.$$

Each cofactor is essentially a subdeterminant of M ; it is a sum of products of elements of the matrix M (some with minus signs attached). Therefore, if the matrix elements are integers, so are the cofactors; and if the determinant is ± 1 , so are the matrix elements of M^{-1} .

6. *(25 pts.)* Let $\vec{F}(\vec{x}) = x\hat{\imath} - y\hat{\jmath} + \hat{k}$ for all $\vec{x}$ in $\mathbf{R}^3$.

(a) Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ when S is the sphere of radius 2 centered at $(1, 0, 0)$ (with outward normal vector).

Easy way: Since the sphere is a closed surface, the first thing we do is check whether Gauss's theorem can be applied. We see that $\nabla \cdot \vec{F} = 1 - 1 + 0 = 0$, which is the best possible circumstance for Gauss's theorem! The integral is equal to 0.

Hard way: I use the spherical coordinate notation in which the polar angle is θ . Since our sphere is not at the origin, we need to write

$$x - 1 = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

$$\begin{aligned}\vec{F} \cdot \hat{n} &= (1 + 2\sin\theta\cos\phi, -2\sin\theta\sin\phi, 1) \cdot (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ &= \sin\theta\cos\phi + 2\sin^2\theta\cos^2\phi - 2\sin^2\theta\sin^2\phi + \cos\theta \\ &= \sin\theta\cos\phi + 2\sin^2\theta\cos(2\phi) + \cos\theta.\end{aligned}$$

This needs to be multiplied by the surface area element $4\sin\theta$ and integrated over θ from 0 to π and over ϕ from 0 to 2π . With modest effort you can see that the integral of each term is 0 .

Even harder way: Write $\iint F_x\,dy\,dz + \cdots$ (cf. (c)) and reduce each term to an integral over θ and ϕ as described in Sec. 7.6.

(b) Is there a vector field $\vec{A}(\vec{x})$ such that $\vec{F} = \nabla \times \vec{A}$?

Yes, since $\nabla \cdot \vec{F} = 0$ everywhere (as already calculated).

(c) Evaluate $\iint_{\tilde{S}} \vec{F} \cdot d\vec{S}$ where $\tilde{S}$ is the surface described by

$$z = x^2 + y^2, \qquad 0 < x < 1, \quad 0 < y < 1$$

(with upward normal vector).

Since this surface is not closed, Gauss's theorem is not (directly) applicable. So we must do this the hard way. This time the hardest way in (a) turns out to be the easiest way. (That is why it pays to know all methods.) Write

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint (x\, dy\, dz - y\, dz\, dx + dx\, dy).$$

Use $dz = 2x\, dx + 2y\, dy$ and $(dx_j)^2 = 0$, $dy\, dx = -dx\, dy$ (or remember Exercise 7.4.5). We get

$$\int_0^1 \int_0^1 (-2x^2 + 2y^2 + 1)\, dx\, dy = 1.$$

(The first two terms cancel by symmetry and the third integral is immediate.)